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# IX. *The Stress produced in a Semi-infinite Solid by Pressure on Part of the Boundary.*

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## *Introduction.*

This paper had its origin in an attempt to throw some light on the important technical question of the safety of foundations. The question is idealised in a certain theory that has been developed as part of the mathematical theory of Elasticity. This theory sets out to give an account of the displacement and stress produced in an elastic solid body by pressure applied to part of its surface. The material of the solid is taken to be homogeneous and isotropic, and the solid is taken to be bounded by an infinite plane, and otherwise unlimited. The solid can be taken to represent the ground on which a building is raised, the pressed areas to represent the bases of walls or pillars, and the pressures to represent the weights supported on such bases. The law of distribution of pressure on the bases of walls and pillars is not known, but it would seem to be reasonable to assume that it is often not very far from being uniform, and thus the special cases of uniform pressure over rectangular and circular areas seem to be of considerable importance.

A formal general solution* of the problem has been known for a long time. It is applicable to any form of boundary of the pressed area, and to any law of distribution of pressure over the area. In this solution the components of displacement, and the components of stress, at any point in the solid, are expressed in terms of the space derivatives of a certain function, called by Boussinesq "le potentiel logarithmique à trois variables." This function is defined as a certain double integral taken over the pressed area. The difficulty of evaluating the integral has been a serious obstacle to the development of the formal solution in special cases. An alternative method of solution† has been devised. This is applicable to a circular boundary only, and has so far been developed only in cases where the pressure is distributed symmetrically about

* The solution is due to J. Boussinesq, who gave it in a series of papers published in Paris, 'Comptes Rendus' for the years 1878–1883, and later developed the theory in a treatise "Applications des potentiels à l'étude de l'équilibre . . . des solides élastiques . . .," Paris, 1885. An account of the theory is given by K. Pearson in Todhunter and Pearson's "History of the Theory of Elasticity," vol. 2, Part 2, pp. 237 *et seq.*, Cambridge, 1893.

† H. Lamb, "On Boussinesq's problem," 'London Math. Soc. Proc.,' vol. 34, p. 276 (1902), and K. Terazawa, "On the elastic equilibrium of a semi-infinite solid . . .," 'Tokyo J. Coll. Sci.,' vol. 37, art. 7, 1916.

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the centre. In this form of solution the components of displacement, and the components of stress, are expressed in terms of single integrals involving BESSEL's functions. The two methods may be referred to as the "potential method" and the "BESSEL's function method."

In this paper the potential method will be developed beyond the stage to which it has been carried previously, and this method will be applied to the important special problems of rectangular and circular areas under uniform pressure. Certain difficulties in regard to finiteness and determinacy of stress present themselves, and a method of evading them, by supposing that the applied pressure is not strictly uniform, will be illustrated by a discussion of the stress produced by some particular distributions of variable pressure. By means of an arithmetical examination of the results obtained, an attempt will be made to furnish material for forming an estimate of strength.

## 1. *Development of the Potential Method.*

§ 1.1. To fix ideas we think of the plane boundary of the semi-infinite solid as horizontal and of the solid as below it. We take this plane to be the plane $z = 0$ of a cartesian co-ordinate system, and the positive sense of the axis of $z$ to be downwards, so that $z$ is positive at every point within the solid. We take $x, y, z$ to be the co-ordinates of a point within the solid, $x', y', 0$ to be those of a point on the plane boundary, and denote the distance between these points by $r$, so that

$$r^2 = (x - x')^2 + (y - y')^2 + z^2, \quad \ldots \ldots \ldots \quad (1)$$

and $r$ is positive. If $(x', y', 0)$ lies inside a certain closed curve, the boundary of the pressed area, we take the applied pressure at this point to be $p$, so that $p$ is a certain function of $x', y'$ defined for points inside a certain curve. BOUSSINESQ's three-dimensional logarithmic potential $\chi$ is then defined by the equation

$$\chi = \iint p \log (z + r)\, dx'\, dy', \quad \ldots \ldots \ldots \quad (2)$$

where the integration is taken over the pressed area.

At any point in the region $z > 0$ the function $\chi$ is continuous and has continuous derivatives, of all orders, with respect to $x, y, z$. All such derivatives can be formed by differentiating under the sign of integration. In the same region all such derivatives tend to zero at infinite distances from the origin.

It is convenient to introduce also the function V, defined by the equation

$$V = \iint pr^{-1}\, dx'\, dy', \quad \ldots \ldots \ldots \quad (3)$$

where the integration is taken over the pressed area. Since V is the Newtonian potential of a surface distribution, its properties are well known.

We have the equations

$$\nabla^2\chi = 0, \quad \nabla^2 V = 0, \quad \frac{\partial\chi}{\partial z} = V, \quad \ldots \ldots \ldots \ldots \quad (4)$$

where $\nabla^2$ denotes the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

The displacement $(u, v, w)$ produced at the point $(x, y, z)$ by the pressure $p$ applied over the pressed area is given by the formulæ*

$$\left.\begin{aligned} u &= -\frac{1}{4\pi}\left(\frac{1}{\lambda+\mu}\frac{\partial\chi}{\partial x} + \frac{z}{\mu}\frac{\partial V}{\partial x}\right), \\ v &= -\frac{1}{4\pi}\left(\frac{1}{\lambda+\mu}\frac{\partial\chi}{\partial y} + \frac{z}{\mu}\frac{\partial V}{\partial y}\right), \\ w &= \frac{1}{4\pi\mu}\left(\frac{\lambda+2\mu}{\lambda+\mu}V - z\frac{\partial V}{\partial z}\right) \end{aligned}\right\}, \quad \ldots \ldots \ldots \ldots \quad (5)$$

in which $\lambda$ and $\mu$ are LAMÉ's elastic constants for the material of the solid.

The corresponding formulæ for the stress-components, deduced from the stress-strain relations of the types

$$\widehat{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}, \quad \widehat{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad \ldots \quad (6)$$

by the aid of equations (4), are

$$\left.\begin{aligned} \widehat{xx} &= \frac{1}{2\pi}\left(\frac{\lambda}{\lambda+\mu}\frac{\partial V}{\partial z} - \frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial x^2} - z\frac{\partial^2 V}{\partial x^2}\right), \\ \widehat{yy} &= \frac{1}{2\pi}\left(\frac{\lambda}{\lambda+\mu}\frac{\partial V}{\partial z} - \frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial y^2} - z\frac{\partial^2 V}{\partial y^2}\right), \\ \widehat{zz} &= \frac{1}{2\pi}\left(\frac{\partial V}{\partial z} - z\frac{\partial^2 V}{\partial z^2}\right), \\ \widehat{yz} &= -\frac{1}{2\pi}z\frac{\partial^2 V}{\partial y\,\partial z}, \quad \widehat{zx} = -\frac{1}{2\pi}z\frac{\partial^2 V}{\partial z\,\partial x}, \\ \widehat{xy} &= -\frac{1}{2\pi}\left(\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial x\,\partial y} + z\frac{\partial^2 V}{\partial x\,\partial y}\right) \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

It follows at once from equations (4) that these formulæ satisfy the stress equations of equilibrium of the type

$$\frac{\partial\widehat{xx}}{\partial x} + \frac{\partial\widehat{xy}}{\partial y} + \frac{\partial\widehat{zx}}{\partial z} = 0, \quad \ldots \ldots \ldots \ldots \quad (8)$$

and from known properties of the potential that they also satisfy the boundary conditions, which express that the part of the plane $z = 0$ which is inside the pressed area

* These formulæ, equivalent to those obtained by Boussinesq, are effectively due to H. HERTZ, "Über die Berührung fester elastischer Körper," 'J. Math.,' (Crelle), vol. 92 (1881), reprinted in "Ges. Werke von Heinrich Hertz," vol. 1, Leipzig, 1895.

is subjected to purely normal pressure $p$, and the rest of this plane is free from traction. All the components of displacement and stress, given by (5) and (7) tend to zero at infinite distances in the region $z > 0$, occupied by the solid, and thus the solution is verified completely.

§ 1.2. It has been stated already that, at points within the solid, the derivatives that are required can be found by differentiating under the sign of integration. For points on the boundary plane, inside the pressed area, or on its bounding curve, such derivatives must be interpreted as limits, to which the values of the space derivatives of $\chi$ or V at $(x, y, z)$, or P, tend, as the point P approaches a point M on the plane $z = 0$. As regards derivatives of V use can be made of known properties of the potential of a surface distribution, but for $\chi$ some general theory is necessary.

If $z$ is changed into $z + h$, where $h$ is a positive constant, $\chi$ is changed into $\chi_1$, where

$$\chi_1 = \iint p \log (z + h + r_1)\, dx'\, dy', \qquad (1)$$

the range of integration is the same as before, and $r_1$ is positive and is given by the equation

$$r_1^2 = (x - x')^2 + (y - y')^2 + (z + h)^2. \qquad (2)$$

Then we have the equation

$$\chi_1 - \chi = \iint p \left\{\int_{-h}^{0} \frac{dz'}{\mathrm{R}}\right\} dx'\, dy', \qquad (3)$$

where R is positive, and is given by the equation

$$\mathrm{R}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2. \qquad (4)$$

Hence we can write

$$\chi = \chi_1 - \chi_2, \qquad (5)$$

where $\chi_2$ is the potential of a volume distribution, viz. :

$$\chi_2 = \iiint p\, \mathrm{R}^{-1}\, dx'\, dy'\, dz', \qquad (6)$$

the "density" $p$ being a function of $x'$, $y'$, independent of $z'$, and the volume of integration being a cylinder or prism, which stands on the pressed area, has generators parallel to the axis of $z$, and is bounded by two transverse sections $z = 0$ and $z = -h$. This volume will be called T. The result can be interpreted in the statement that $\chi$ is the sum of the three-dimensional logarithmic potential $\chi_1$ of the surface distribution $p$, over the projection of the pressed area on a fixed plane $z = -h$, and the Newtonian potential $-\chi_2$ of the volume distribution $-p$ through the volume T.

At any point in the solid, or on its boundary plane, the space derivatives of $\chi_1$, of all orders, can be formed by differentiating under the sign of integration, and have definite finite values. All such derivatives are continuous functions of $x$, $y$, $z$.

§ 1.3. In the region outside T the space derivatives of $\chi_2$ of all orders have definite finite values which may be found by differentiating under the sign of integration. In particular, this statement applies to points on the bounding plane $z = 0$ outside the pressed area.

We consider the derivative $\partial\chi_2/\partial x$, and observe that it can be expressed as the sum of two potentials. Let $l$, $m$, $n$ denote the direction cosines of the outward normal to the surface S bounding the volume T. On the transverse faces $z = 0$ and $z = -h$ we have $l = 0$ and $m = 0$; on the surface between them we have $n = 0$, and $l$, $m$ are there the same as the direction cosines of the normal to the bounding curve $s$ of the pressed area, drawn in its plane and outwards. We shall assume that $p$, as a function of $x'$ and $y'$, is differentiable. We have

$$\frac{\partial\chi_2}{\partial x} = \iiint p\,\frac{\partial \mathrm{R}^{-1}}{\partial x}\,dx'\,dy'\,dz' = -\iiint p\,\frac{\partial \mathrm{R}^{-1}}{\partial x'}\,dx'\,dy'\,dz',$$

or

$$\frac{\partial\chi_2}{\partial x} = \iiint \frac{\partial p}{\partial x'}\,\frac{1}{\mathrm{R}}\,dx'\,dy'\,dz' - \iint lp\,\frac{1}{\mathrm{R}}\,d\mathrm{S}, \quad \ldots \ldots \ldots \quad (1)$$

where the volume integration is taken through the volume T. The right-hand member of (1) is the sum of the potential of a volume distribution $\partial p/\partial x'$ in T and the potential of a surface distribution $-lp$ on S. The derivative $\partial\chi_2/\partial y$ can be transformed in the same way.

It will follow from known properties of the potentials of volume and surface distributions that, as the point P, or $(x, y, z)$, approaches a point M on the plane $z = 0$, each of these derivatives tends to a definite finite limit, and further that each of these limits varies continuously as M crosses the bounding curve of the pressed area.

The second derivative $\partial^2\chi_2/\partial x^2$ can be interpreted, in accordance with (1), as the sum of the $x$-components of attraction due to the volume distribution $\partial p/\partial x'$ in T and the surface distribution $-lp$ on S. It will follow from known properties of the attractions of such distributions that this derivative tends to a definite finite limit as P approaches any point M on the plane $z = 0$, whether M is outside or inside the pressed area. Further it will follow that the limit of the same derivative changes discontinuously as M crosses the bounding curve $s$ of this area, and has no definite finite value at a point on this curve. Similar statements hold for the derivatives $\partial^2\chi_2/\partial y^2$ and $\partial^2\chi_2/\partial x\,\partial y$.

The discontinuity arises from the surface integral in (1). This surface integral can be transformed into a line integral taken round $s$ by the formula

$$\iint lp\mathrm{R}^{-1}\,d\mathrm{S} = \int lp \log\frac{z+h+r_1}{z+r}\,ds, \quad \ldots \ldots \ldots \quad (2)$$

where $r_1$ is given by 1.2 (2). It follows that, if $p = 0$ on $s$, there is no discontinuity. In this case $\partial^2\chi_2/\partial x^2$, $\partial^2\chi_2/\partial y^2$, and $\partial^2\chi_2/\partial x\,\partial y$ tend to definite finite limits as P approaches

M, whether M is inside $s$, or outside it, or on it, and these limits change continuously as M crosses $s$.

§ 1.4. The necessary theorems in regard to V and its derivatives can be stated without proof.

The function V tends to a definite finite limit as P approaches M, whether M is inside $s$, outside $s$, or on $s$. The limit varies continuously as M crosses $s$.

The derivative $\partial V/\partial z$ tends to a definite finite limit as P approaches M, whether M is inside $s$, outside $s$, or on $s$. The limit changes discontinuously as M crosses $s$. The product $z \cdot \partial V/\partial z$ tends to the definite limit zero as P approaches M, whether M is inside $s$, outside $s$, or on $s$.

The derivatives $\partial V/\partial x$ and $\partial V/\partial y$ tend to definite finite limits as P approaches M, whether M is inside $s$ or outside $s$. If M is on $s$ they generally become infinite* at M. But, in every case, the products $z \cdot \partial V/\partial x$ and $z \cdot \partial V/\partial y$ tend to the definite limit zero as P approaches M.

From these remarks, and the theory already given for the first derivatives of the two constituents of $\chi$, it follows that the displacement of the initially plane bounding surface of the solid, as expressed by the formulæ 1.1 (5), has a definite value for any point M on this plane, and that this value changes continuously as M crosses $s$.

Each of the six second derivatives of V tends to a definite finite limit as P approaches M, whether M is inside $s$, or outside $s$. In general these limits change discontinuously as M crosses $s$. When M is on $s$ these derivatives do not in general tend to finite limits. If M is either inside or outside $s$, the product of $z$ and any such derivative tends to zero as P approaches M. If M is on $s$, these products are in general indeterminate, tending to limits as P approaches M, which limits depend upon the direction of the tangent at M to the path of P.

From these remarks, and the theory already given for the second derivatives of the two constituents of $\chi$, it follows that the stress in the solid, as expressed by the formulæ § 1.1 (7), has a definite finite value at any point M on the plane $z = 0$, whether outside or inside $s$, but becomes indeterminate if M is on $s$.

The indeterminacy disappears if $p = 0$ on $s$.

§ 1.5. For the calculation of the stress produced by given pressure $p$ on a prescribed area of the bounding plane it is unnecessary to evaluate the integrals by which the functions V and $\chi$ are defined, and it is often simpler to evaluate the integrals by which the necessary derivatives of these functions are expressed. This is especially the case when $p$ is constant over the pressed area. When this is so, the double integrals can be transformed into line integrals taken round the bounding curve $s$ of the pressed area by means of the theorem

$$\iint\left(\frac{\partial \xi}{\partial x'} + \frac{\partial \eta}{\partial y'}\right) dx'\, dy' = \int (l\xi + m\eta)\, ds, \quad \ldots \ldots \ldots \quad (1)$$

* *Cf.* H. Poincaré, "Théorie du potentiel Newtonien," pp. 121, *et seq.*, Paris, 1899.

for $x$ and $y$ enter into the expressions for V and $\chi$ only in the combinations $x - x'$ and $y - y'$. For example, $\partial^2\chi/\partial x^2$ and $\partial^2\chi/\partial y^2$ can be transformed in this way, and then $\partial V/\partial z$ can be obtained from the formula

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial V}{\partial z} = 0, \quad \ldots \ldots \quad (2)$$

which follows from § 1.1 (4). Again $\partial^2 V/\partial x^2$ and $\partial^2 V/\partial y^2$ can be transformed in this way, and then $\partial^2 V/\partial z^2$ can be obtained from the equation $\nabla^2 V = 0$. It appears that all the double integrals that enter into the expressions § 1.1 (7) can thus be transformed into single integrals.

When $p$ is constant, the equation 1.1 (3) gives

$$\frac{\partial V}{\partial z} = - p\ \Omega, \quad \ldots \ldots \quad (3)$$

where

$$\Omega = \iint \frac{z}{r^3}\, dx'\, dy', \quad \ldots \ldots \quad (4)$$

so that $\Omega$ is the solid angle subtended by the pressed area at the point $(x, y, z)$. This remark gives a simple geometrical interpretation of the derivative $\partial V/\partial z$, and may in some cases lead to a method of finding this derivative.

For the calculation of the displacement it is necessary to determine V, so that it may easily happen that the displacement is more difficult to calculate then the stress. It is never necessary to determine $\chi$.

Although the final result of § 1.4 renders it unlikely that the hypothesis of uniform pressure over the pressed area can be rigorously exact, nevertheless it seems to be worth while, for reasons stated in the Introduction, to work out the stress distributions due to uniform pressure on some areas with simple bounding curves.

## 2. *Uniform Pressure over Rectangle.*

§ 2.1. Let the boundary of the pressed area be given by the equations

$$z = 0, \quad x = \pm a, \quad y = \pm b. \quad \ldots \ldots \quad (1)$$

In the figure (fig. 1) let the sides AB, CD be of length $2a$, and the sides BC, DA of length $2b$, and let $M_2$, $M_1$ be the middle points of BC, DA. The sense of description indicated by the order of the letters, A, B, C, D, is the positive (right-handed) sense, the positive sense of the axis of $z$ being downwards.

It will be convenient to denote the distances of the point $(x, y, z)$ in the solid from points on the lines DA and BC respectively by $r_{1,0}$ and $r_{2,0}$, so that

$$r_{1,0}^2 = (a - x)^2 + (y' - y)^2 + z^2, \quad r_{2,0}^2 = (a + x)^2 + (y' - y)^2 + z^2. \quad . \quad (2)$$

Further it will be convenient to denote the distances of $(x, y, z)$ from the corners A, B, C, D, respectively by $a_1$, $b_2$, $c_3$, $d_4$.

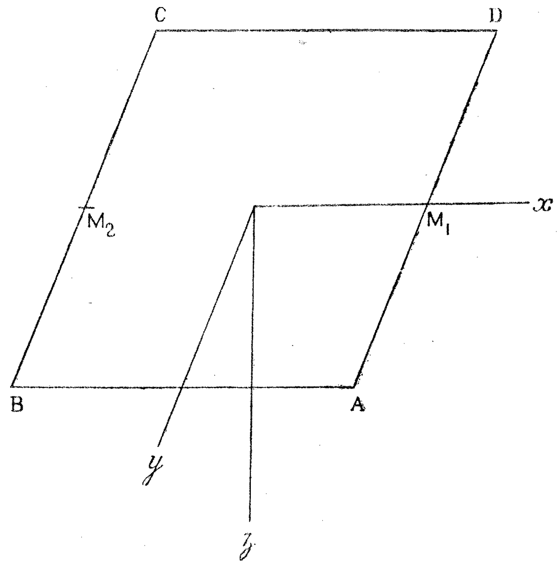


FIG. 1.

§ 2.2. We begin by transforming the double integral expressing $\partial^2\chi/\partial x^2$. Since $p$ is constant, we have

$$\frac{1}{p}\frac{\partial^2\chi}{\partial x^2} = \int_{-b}^{b}\int_{-a}^{a}\frac{\partial^2}{\partial x'^2}\{\log(z+r)\}\,dx'\,dy'$$

$$= \int_{-b}^{b}\int_{-a}^{a}\frac{\partial}{\partial x'}\left\{\frac{x'-x}{r(z+r)}\right\}dx'\,dy',$$

or

$$\frac{1}{p}\frac{\partial^2\chi}{\partial x^2} = \int_{-b}^{b}\left\{\frac{a-x}{r_{1,0}(z+r_{1,0})}+\frac{a+x}{r_{2,0}(z+r_{2,0})}\right\}dy'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now

$$\frac{1}{r_{1,0}(z+r_{1,0})} = \frac{1}{{r_{1,0}}^2-z^2}\left(1-\frac{z}{r_{1,0}}\right) = \frac{1}{(a-x)^2+(y'-y)^2}\left(1-\frac{z}{r_{1,0}}\right). \quad . \quad . \quad (2)$$

If we put temporarily

$$(a-x)^2+z^2=\beta^2, \quad \beta>0, \quad y'-y=\beta\tan\theta, \quad r_{1,0}=\beta\sec\theta, \quad . \quad . \quad . \quad (3)$$

we get

$$\frac{dy'}{\{(a-x)^2+(y'-y)^2\}\,r_{1,0}} = \frac{\sec\theta\,d\theta}{(a-x)^2+\beta^2\tan^2\theta} = \frac{\cos\theta\,d\theta}{(a-x)^2\cos^2\theta+\beta^2\sin^2\theta}$$

$$= \frac{\cos\theta\,d\theta}{(a-x)^2+z^2\sin^2\theta} = \frac{1}{z\,|a-x|}\,d\tan^{-1}\frac{z\sin\theta}{|a-x|}$$

$$= \frac{1}{z\,|a-x|}\,d\tan^{-1}\frac{z\,(y'-y)}{|a-x|\,r_{1,0}}.$$

Similar transformations can be made of the term of (1) that contains $r_{2,0}$, and thus we can evaluate the integrals in (1) and obtain the formula

$$\frac{1}{p}\frac{\partial^2\chi}{\partial x^2} = \frac{a-x}{|a-x|}\left\{\tan^{-1}\frac{b-y}{|a-x|} + \tan^{-1}\frac{b+y}{|a-x|} - \tan^{-1}\frac{z(b-y)}{|a-x|\,a_1} - \tan^{-1}\frac{z(b+y)}{|a-x|\,d_4}\right\}$$
$$+ \frac{a+x}{|a+x|}\left\{\tan^{-1}\frac{b-y}{|a+x|} + \tan^{-1}\frac{b+y}{|a+x|} - \tan^{-1}\frac{z(b-y)}{|a+x|\,b_2} - \tan^{-1}\frac{z(b+y)}{|a+x|\,c_3}\right\}, \tag{4}$$

in which it is clear that all the inverse tangents have values between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. In the above the upright lines indicate, as usual, that the absolute value, without regard to sign, of the expression included between them is to be taken. Observing that $(a-x)/|a-x|$ is $+1$ when $a > x$ and $-1$ when $a < x$, we see that (4) can be replaced by

$$\frac{\partial^2\chi}{\partial x^2} = p\left[\tan^{-1}\frac{b-y}{a-x} + \tan^{-1}\frac{b+y}{a-x} - \tan^{-1}\frac{z(b-y)}{(a-x)\,a_1} - \tan^{-1}\frac{z(b+y)}{(a-x)\,d_4}\right.$$
$$\left. + \tan^{-1}\frac{b-y}{a+x} + \tan^{-1}\frac{b+y}{a+x} - \tan^{-1}\frac{z(b-y)}{(a+x)\,b_2} - \tan^{-1}\frac{z(b+y)}{(a+x)\,c_3}\right], \tag{5}$$

in which all the inverse tangents have values between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

We can write down the corresponding expression for $\partial^2\chi/\partial y^2$ in the form

$$\frac{\partial^2\chi}{\partial y^2} = p\left[\tan^{-1}\frac{a-x}{b-y} + \tan^{-1}\frac{a+x}{b-y} - \tan^{-1}\frac{z(a-x)}{(b-y)\,a_1} - \tan^{-1}\frac{z(a+x)}{(b-y)\,b_2}\right.$$
$$\left. + \tan^{-1}\frac{a-x}{b+y} + \tan^{-1}\frac{a+x}{b+y} - \tan^{-1}\frac{z(a-x)}{(b+y)\,d_4} - \tan^{-1}\frac{z(a+x)}{(b+y)\,c_3}\right], \tag{6}$$

in which again all the inverse tangents have values between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

§ 2.3. The expressions in the right-hand members of § 2.2 (5) and (6) admit of simple geometrical interpretations. If, for simplicity, we take the case where the projection Q of the point P, or $(x, y, z)$, falls within the rectangle, it can be proved easily that the expression

$$\tan^{-1}\frac{b-y}{a-x} - \tan^{-1}\frac{z(b-y)}{(a-x)\,a_1}$$

is equal to the excess of the angle between the planes PAD, PAQ over the angle between the lines AD, AQ. See fig. 2. The other pairs of inverse tangents admit of similar interpretations.

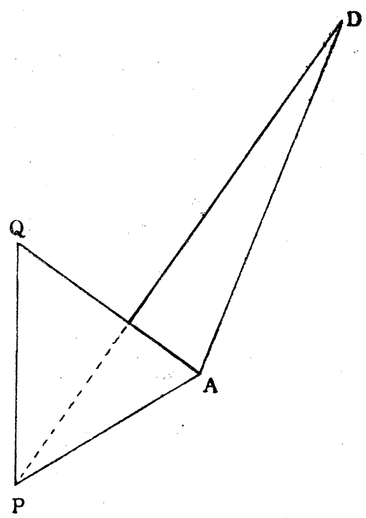

FIG. 2.

It is to be noted that the expressions become indeterminate at points on the rectangular boundary. For example, at a point on the side $z = 0$, $x = a$, where $b > y > -b$, the distance $a_1$ is $b - y$, and, if the point $(x, y, z)$ approaches a point on this side of the rectangle, the expression

$$\tan^{-1} \frac{z(b-y)}{(a-x)\,a_1}$$

does not tend to a definite limit. This indeterminacy is in accord with the general theory of § 1.3.

§ 2.4. From the results § 2.2 (5) and § 2.2. (6), combined with § 1.5 (2) and § 1.5 (3), we obtain an expression for the solid angle $\Omega$, subtended by the rectangle at the point $(x, y, z)$, as a sum of sixteen inverse tangents. The arguments of eight of these are reciprocals in pairs, so that their sum is a multiple of $\frac{1}{2}\pi$. If the case where $a > x > -a$ and $b > y > -b$ is chosen for simplicity, this sum is $2\pi$.

The sum of the third inverse tangents in § 2.2 (5) and § 2.2. (6) is

$$\tan^{-1} \frac{za_1\{(a-x)^2+(b-y)^2\}}{(a-x)(b-y)(a_1^2-z^2)},$$

or

$$\tan^{-1} \frac{za_1}{(a-x)(b-y)},$$

which is the same as

$$\cos^{-1} \frac{(a-x)(b-y)}{\sqrt{\{(a-x)^2+z^2\}}\sqrt{\{(b-y)^2+z^2\}}}.$$

In the case chosen we shall have therefore

$$\Omega = 2\pi - \cos^{-1}\frac{(a-x)(b-y)}{\sqrt{\{(a-x)^2+z^2\}}\sqrt{\{(b-y)^2+z^2\}}} - \cos^{-1}\frac{(a-x)(b+y)}{\sqrt{\{(a-x)^2+z^2\}}\sqrt{\{(b+y)^2+z^2\}}}$$
$$-\cos^{-1}\frac{(a+x)(b-y)}{\sqrt{\{(a+x)^2+z^2\}}\sqrt{\{(b-y)^2+z^2\}}} - \cos^{-1}\frac{(a+x)(b+y)}{\sqrt{\{(a+x)^2+z^2\}}\sqrt{\{(b+y)^2+z^2\}}}. \quad (1)$$

This formula could be obtained by elementary geometrical methods, and, if all the inverse cosines are taken to have values between 0 and $\pi$, it holds for all positions of the point $(x, y, z)$.

The derivative $\partial V/\partial z$ is given by the equation § 1.5 (3), viz. :

$$\frac{\partial V}{\partial z} = -p\,\Omega,$$

where $\Omega$ is given by (1).

§ 2.5. The remaining derivatives can be obtained without difficulty.

We have at once

$$\frac{1}{p}\frac{\partial^2\chi}{\partial x\,\partial y} = \int_{-b}^{b}\int_{-a}^{a} \frac{\partial^2}{\partial x'\,\partial y'}\{\log(z+r)\}\,dx'\,dy',$$

giving

$$\frac{\partial^2\chi}{\partial x\,\partial y} = p\log\frac{(z+a_1)(z+c_3)}{(z+b_2)(z+d_4)} \quad \ldots\ldots\ldots\ldots \quad (1)$$

This expression becomes infinite at the corners of the rectangle—a matter which will be considered presently.

Again we have

$$\frac{1}{p}\frac{\partial^2 V}{\partial x^2} = \int_{-b}^{b}\int_{-a}^{a} -\frac{\partial}{\partial x'}\left(\frac{x'-x}{r^3}\right) dx'\,dy',$$

giving

$$\frac{1}{p}\frac{\partial^2 V}{\partial x^2} = -\int_{-b}^{b}\left(\frac{a-x}{r_{1,0}{}^3} + \frac{a+x}{r_{2,0}{}^3}\right) dy'. \quad \ldots \ldots \ldots \ldots \quad (2)$$

In the notation of § 2.2 (3)

$$\frac{dy'}{r_{1,0}{}^3} = \frac{1}{\beta^2}\cos\theta\, d\theta = \frac{1}{\beta^2}\, d\left(\frac{y'-y}{r_{1,0}}\right),$$

and we have therefore

$$\frac{\partial^2 V}{\partial x^2} = -p\left\{\frac{a-x}{(a-x)^2+z^2}\left(\frac{b-y}{a_1} + \frac{b+y}{d_4}\right) + \frac{a+x}{(a+x)^2+z^2}\left(\frac{b-y}{b_2} + \frac{b+y}{c_3}\right)\right\}. \quad (3)$$

In the same way it can be proved that

$$\frac{\partial^2 V}{\partial y^2} = -p\left\{\frac{b-y}{(b-y)^2+z^2}\left(\frac{a-x}{a_1} + \frac{a+x}{b_2}\right) + \frac{b+y}{(b+y)^2+z^2}\left(\frac{a-x}{d_4} + \frac{a+x}{c_3}\right)\right\}. \quad (4)$$

These expressions become infinite at points on the rectangular boundary, and the products of them and $z$ do not tend to definite limits as the point $(x, y, z)$ approaches a point on this boundary. This result is in accord with the theory of § 1.4.

From the equations (3) and (4), combined with the equation $\nabla^2 V = 0$, it is found that

$$\frac{\partial^2 V}{\partial z^2} = p\left\{\frac{a-x}{(a-x)^2+z^2}\left(\frac{b-y}{a_1} + \frac{b+y}{d_4}\right) + \frac{a+x}{(a+x)^2+z^2}\left(\frac{b-y}{b_2} + \frac{b+y}{c_3}\right)\right.$$
$$\left. + \frac{b-y}{(b-y)^2+z^2}\left(\frac{a-x}{a_1} + \frac{a+x}{b_2}\right) + \frac{b+y}{(b+y)^2+z^2}\left(\frac{a-x}{d_4} + \frac{a+x}{c_3}\right)\right\}. \quad (5)$$

Again we have

$$\frac{1}{p}\frac{\partial^2 V}{\partial x\,\partial z} = \int_{-b}^{b}\int_{-a}^{a}\frac{\partial}{\partial x'}\left(\frac{z}{r^3}\right) dx'\,dy',$$

giving

$$\frac{1}{p}\frac{\partial^2 V}{\partial x\,\partial z} = z\int_{-b}^{b}\left(\frac{1}{r_{1,0}{}^3} - \frac{1}{r_{2,0}{}^3}\right) dy', \quad \ldots \ldots \ldots \ldots \quad (6)$$

from which it follows, as in previous work, that

$$\frac{\partial^2 V}{\partial x\,\partial z} = p\left\{\frac{z}{(a-x)^2+z^2}\left(\frac{b-y}{a_1} + \frac{b+y}{d_4}\right) - \frac{z}{(a+x)^2+z^2}\left(\frac{b-y}{b_2} + \frac{b+y}{c_3}\right)\right\} \quad (7)$$

In like manner it is found that

$$\frac{\partial^2 V}{\partial y\,\partial z} = p\left\{\frac{z}{(b-y)^2+z^2}\left(\frac{a-x}{a_1}+\frac{a+x}{b_2}\right)-\frac{z}{(b+y)^2+z^2}\left(\frac{a-x}{d_4}+\frac{a+x}{c_3}\right)\right\} \quad (8)$$

The expressions in (5), (7) and (8) become infinite at points on the rectangular boundary, and the products of these expressions and $z$ do not tend to definite limits as $(x, y, z)$ approaches such a point.

Finally we have

$$\frac{1}{p}\frac{\partial^2 V}{\partial x\,\partial y} = \int_{-b}^{b}\int_{-a}^{a}\frac{\partial^2}{\partial x'\,\partial y'}\left(\frac{1}{r}\right)dx'\,dy',$$

giving

$$\frac{\partial^2 V}{\partial x\,\partial y} = p\left(\frac{1}{a_1}-\frac{1}{b_2}+\frac{1}{c_3}-\frac{1}{d_4}\right). \quad (9)$$

This expression becomes infinite at the corners of the rectangle, and the product of the expression and $z$ does not tend to a definite limit as $(x, y, z)$ approaches such a point.

All the double integrals that occur in the expressions § 1.1 (7) for the stress-components have now been evaluated.

§ 2.6. We consider the case of a very long rectangle, and suppose that the ratio $b : a$ is very great, and we calculate the stress-components at a point in the middle transverse section $y = 0$. We shall suppose that $x$ and $z$, as well as $a$, are small compared with $b$. Then it is sufficient to take $x$ to be positive, and it is appropriate to introduce two angles $\theta_1$ and $\theta_2$, such that $\pi > \theta_1 > 0$ and $\frac{1}{2}\pi > \theta_2 > 0$, by the equations

$$\tan\theta_1 = \frac{z}{x-a}, \qquad \tan\theta_2 = \frac{z}{x+a}. \quad (1)$$

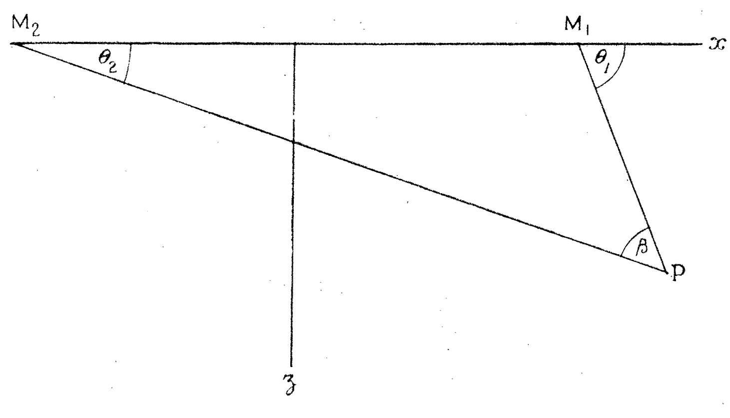


Fig. 3.

These angles are shown in fig. 3, where P is a point $(x, 0, z)$ for which $x$ and $z$ are positive, and $M_1$, $M_2$ are the middle points of the sides DA, BC of the rectangle, as shown in fig. 1 of § 2.1.

The limiting values of the relevant derivatives, as $b/a \to \infty$, are as follows :—

$$\left.\begin{aligned} &\frac{\partial^2\chi}{\partial x^2} = 2p(\theta_1 - \theta_2), \quad \frac{\partial^2\chi}{\partial y^2} = 0, \quad \frac{\partial^2\chi}{\partial x\,\partial y} = 0, \quad \frac{\partial V}{\partial z} = -2p(\theta_1 - \theta_2), \\ &\frac{\partial^2 V}{\partial x^2} = \frac{p}{z}(\sin 2\theta_1 - \sin 2\theta_2), \quad \frac{\partial^2 V}{\partial y^2} = 0, \quad \frac{\partial^2 V}{\partial x\,\partial y} = 0, \\ &\frac{\partial^2 V}{\partial x\,\partial z} = \frac{2p}{z}(\sin^2\theta_1 - \sin^2\theta_2), \quad \frac{\partial^2 V}{\partial y\,\partial z} = 0, \quad \frac{\partial^2 V}{\partial z^2} = -\frac{p}{z}(\sin 2\theta_1 - \sin 2\theta_2). \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

The limiting values of the stress-components that do not vanish are

$$\left.\begin{aligned} \widehat{xx} &= -\frac{p}{2\pi}\{2(\theta_1 - \theta_2) + (\sin 2\theta_1 - \sin 2\theta_2)\}, \\ \widehat{zz} &= -\frac{p}{2\pi}\{2(\theta_1 - \theta_2) - (\sin 2\theta_1 - \sin 2\theta_2)\}, \\ \widehat{zx} &= -\frac{p}{\pi}(\sin^2\theta_1 - \sin^2\theta_2). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The solid is in a state of plane stress. When the magnitudes of the principal stresses and the directions of the lines of stress are determined in the usual way, it is found that the directions in question are the bisectors of the angle $M_1PM_2$, so that the lines of stress are ellipses and hyperbolas having $M_1$ and $M_2$ as foci. The principal stress, which is pressure (not tension), along the normal to the ellipse of the family that passes through a point P, is of magnitude

$$\frac{p}{\pi}(\beta + \sin\beta),$$

and the principal stress, which is pressure, along the tangent to the hyperbola of the family that passes through P, is of magnitude

$$\frac{p}{\pi}(\beta - \sin\beta),$$

where $\beta$ denotes the angle $M_1PM_2$, or $\theta_1 - \theta_2$. Both being compressive, there is nowhere any tensile stress. Both principal stresses have constant values along any circular arc, passing through $M_1$ and $M_2$, and having its centre on the axis of $z$.

These results were obtained by J. H. Michell,* without any reference to Boussinesq's theory, by interpreting the formula for the stress-function (in plane stress) due to two simple singularities on a straight boundary. The theory has been discussed further by E. G. Coker,† and may be regarded as indicating the state of stress in the foundation near the middle transverse section of a long wall.

* 'London Math. Soc. Proc.,' vol. 34, p. 134 (1902).
† 'London, Inst. Mech. Engineers' Proc.' (1921).

### 3. *Uniform Pressure over Circle.*

§ 3.1. When the boundary of the pressed area is a circle, and the pressure is uniform, or is distributed symmetrically about the centre, it is appropriate to introduce cylindrical co-ordinates $\rho$, $z$, $\omega$, so that

$$x = \rho \cos \omega, \qquad y = \rho \sin \omega. \qquad \qquad (1)$$

Then we write also

$$x' = \rho' \cos \omega', \qquad y' = \rho' \sin \omega', \qquad \qquad (2)$$

and note that

$$r^2 = \rho^2 + \rho'^2 + z^2 - 2\rho\rho' \cos(\omega' - \omega). \qquad \qquad (3)$$

The functions V and $\chi$, and their differential coefficients (of all orders) with respect to $\rho$ and $z$, are independent of $\omega$.

If $u_\rho$ and $u_z$ denote the components of the displacement in the directions of increase of $\rho$ and $z$, the formulæ § 1.1 (5) for the displacement can be replaced by

$$\left.\begin{aligned} u_\rho &= -\frac{1}{4\pi}\left(\frac{1}{\lambda+\mu}\frac{\partial\chi}{\partial\rho} + \frac{z}{\mu}\frac{\partial V}{\partial\rho}\right), \\ u_z &= \frac{1}{4\pi\mu}\left(\frac{\lambda+2\mu}{\lambda+\mu}V - z\frac{\partial V}{\partial z}\right) \end{aligned}\right\} \qquad \qquad (4)$$

The formulæ § 1.1 (7) for the stress can be replaced by

$$\left.\begin{aligned} \widehat{\rho\rho} &= \frac{1}{2\pi}\left(\frac{\lambda}{\lambda+\mu}\frac{\partial V}{\partial z} - \frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial\rho^2} - z\frac{\partial^2 V}{\partial\rho^2}\right), \\ \widehat{\omega\omega} &= \frac{1}{2\pi}\left(\frac{\lambda}{\lambda+\mu}\frac{\partial V}{\partial z} - \frac{\mu}{\lambda+\mu}\frac{1}{\rho}\frac{\partial\chi}{\partial\rho} - \frac{z}{\rho}\frac{\partial V}{\partial\rho}\right), \\ \widehat{zz} &= \frac{1}{2\pi}\left(\frac{\partial V}{\partial z} - z\frac{\partial^2 V}{\partial z^2}\right), \\ \widehat{\rho z} &= -\frac{z}{2\pi}\frac{\partial^2 V}{\partial\rho\,\partial z}, \quad \widehat{\omega\rho} = 0, \quad \widehat{\omega z} = 0 \end{aligned}\right\} \qquad \qquad (5)$$

A complete solution of the problem for uniform pressure over a circle was obtained by K. Terazawa (*op. cit. ante*) by the Bessel's function method. He determined the displacement and the stress. It appears to be desirable to apply the potential method.

§ 3.2. We take $p$ to be constant. We denote the radius of the circle bounding the pressed area by $a$, so that $\rho' = a$ at this boundary.

To evaluate $\partial\chi/\partial\rho$ we observe that this derivative is the value of $\partial\chi/\partial x$ at a point $(\rho, z)$ in the plane $\omega = 0$. Now we have

$$\frac{1}{p}\frac{\partial\chi}{\partial x} = \iint -\frac{\partial\{\log(z+r)\}}{\partial x'}\,dx'\,dy', \qquad \qquad (1)$$

where the integral is taken over the area within the circle $\rho' = a$. Since, in the notation of § 1.5, $ds = a\,d\omega'$ and $l = \cos\omega'$, this equation can be transformed into

$$\frac{1}{p}\frac{\partial\chi}{\partial x} = \int_0^{2\pi} -\cos\omega' . \log(z + r')\, a\, d\omega', \quad \ldots \quad (2)$$

where

$$r'^2 = \rho^2 + a^2 + z^2 - 2\rho a \cos(\omega' - \omega), \quad \ldots \quad (3)$$

and $r'$ is positive. After an integration by parts (2) becomes

$$\frac{1}{p}\frac{\partial\chi}{\partial x} = a\int_0^{2\pi} \sin\omega' \frac{\partial\{\log(z + r')\}}{\partial\omega'}\, d\omega',$$

or

$$\frac{1}{p}\frac{\partial\chi}{\partial x} = a^2\rho \int_0^{2\pi} \sin\omega' \sin(\omega' - \omega) \frac{1}{r'(z + r')} d\omega'. \quad \ldots \quad (4)$$

Hence we have

$$\frac{1}{p}\frac{\partial\chi}{\partial\rho} = a^2\rho \int_0^{2\pi} \frac{\sin^2\omega'}{R(z + R)}\, d\omega', \quad \ldots \quad (5)$$

where

$$R^2 = \rho^2 + a^2 + z^2 - 2\rho a \cos\omega', \quad \ldots \quad (6)$$

and R is positive.

Now

$$\frac{1}{R(z + R)} = \frac{1}{R^2 - z^2}\left(1 - \frac{z}{R}\right),$$

and therefore (5) becomes

$$\frac{1}{p}\frac{\partial\chi}{\partial\rho} = a^2\rho \int_0^{2\pi} \frac{\sin^2\omega'}{a^2 + \rho^2 - 2\rho a \cos\omega'}\left(1 - \frac{z}{R}\right) d\omega', \quad \ldots \quad (7)$$

where R is given by (6).

The analysis necessary for evaluating the integral in (7) is rather long. The result will be found in § 3.24 below.

§ 3.21. We have the identity

$$\sin^2\omega' = -\frac{(a^2 + \rho^2 - 2a\rho\cos\omega')^2}{4a^2\rho^2} + \frac{(a^2 + \rho^2)(a^2 + \rho^2 - 2a\rho\cos\omega')}{2a^2\rho^2} - \frac{(a^2 - \rho^2)^2}{4a^2\rho^2}, \quad (1)$$

and therefore

$$\int_0^{2\pi} \frac{\sin^2\omega'\, d\omega'}{a^2 + \rho^2 - 2a\rho\cos\omega'} = -\frac{\pi(a^2 + \rho^2)}{2a^2\rho^2} + \frac{\pi(a^2 + \rho^2)}{a^2\rho^2} - \frac{(a^2 - \rho^2)^2}{4a^2\rho^2}\int_0^{2\pi} \frac{d\omega'}{a^2 + \rho^2 - 2a\rho\cos\omega'}; \quad . \quad (2)$$

and, as may be proved easily,

$$\int_0^{2\pi} \frac{d\omega'}{a^2 + \rho^2 - 2a\rho\cos\omega'} = \frac{2\pi}{|a^2 - \rho^2|}. \quad \ldots \quad (3)$$

Hence

$$\int_0^{2\pi} \frac{\sin^2 \omega' \, d\omega'}{a^2 + \rho^2 - 2a\rho \cos \omega'} = \pi \frac{(a^2 + \rho^2) - |a^2 - \rho^2|}{2a^2\rho^2}, \quad \ldots \ldots \quad (4)$$

so that it is $\pi/a^2$ when $\rho < a$, and $\pi/\rho^2$ when $\rho > a$.

It remains to evaluate

$$\int_0^{2\pi} \frac{\sin^2 \omega'}{a^2 + \rho^2 - 2a\rho \cos \omega'} \frac{d\omega'}{\sqrt{(a^2 + \rho^2 + z^2 - 2a\rho \cos \omega')}} = \mathrm{I}, \text{ say.}$$

By means of the identity (1) we can write

$$\mathrm{I} = -\frac{1}{4a^2\rho^2} \mathrm{I}_1 + \frac{a^2 + \rho^2}{2a^2\rho^2} \mathrm{I}_2 - \frac{(a^2 - \rho^2)^2}{4a^2\rho^2} \mathrm{I}_3, \quad \ldots \ldots \quad (5)$$

where

$$\left.\begin{aligned} \mathrm{I}_1 &= \int_0^{2\pi} \frac{a^2 + \rho^2 - 2a\rho \cos \omega'}{\sqrt{(a^2 + \rho^2 + z^2 - 2a\rho \cos \omega')}} d\omega' \\ \mathrm{I}_2 &= \int_0^{2\pi} \frac{1}{\sqrt{(a^2 + \rho^2 + z^2 - 2a\rho \cos \omega')}} d\omega' \\ \mathrm{I}_3 &= \int_0^{2\pi} \frac{1}{(a^2 + \rho^2 - 2a\rho \cos \omega')\sqrt{(a^2 + \rho^2 + z^2 - 2a\rho \cos \omega')}} d\omega' \end{aligned}\right\} \quad . \quad . \quad (6)$$

These are elliptic integrals. In the process of reduction of them to normal forms it is convenient to introduce two lengths $r_1$ and $r_2$, which are the distances of the point $(\rho, z)$ in the plane $\omega = 0$ from the points $\mathrm{M}_1$, $\mathrm{M}_2$ on the bounding circle that are nearest to it and farthest from it. These lengths are given by the equations

$$r_1^2 = (a - \rho)^2 + z^2, \qquad r_2^2 = (a + \rho)^2 + z^2, \quad \ldots \ldots \quad (7)$$

and $r_1$ and $r_2$ are positive. Then we have

$$a^2 + \rho^2 + z^2 - 2a\rho \cos \omega' = r_2^2 - (r_2^2 - r_1^2) \cos^2 \tfrac{1}{2}\omega'. \quad \ldots \ldots \quad (8)$$

It is seen that the elliptic integrals are formed with the modulus $\sqrt{(1 - r_1^2/r_2^2)}$, and it turns out to be most convenient to treat this as the complementary modulus $k'$ and write

$$k = \frac{r_1}{r_2}. \quad \ldots \ldots \quad (9)$$

On putting

$$2\phi_1 = \pi - \omega', \quad \ldots \ldots \quad (10)$$

it is seen that

$$\mathrm{I}_2 = \frac{4}{r_2} \int_0^{\frac{1}{2}\pi} \frac{d\phi_1}{\sqrt{(1 - k'^2 \sin^2 \phi_1)}},$$

or

$$\mathrm{I}_2 = \frac{4}{r_2} \mathrm{K}' \quad \ldots \ldots \quad (11)$$

where K′ denotes, as usual, the complete elliptic integral of the first kind with modulus $k'$.

By means of the same transformations it is found that

$$I_1 = 4r_2 \int_0^{\frac{1}{2}\pi} \sqrt{(1 - k'^2 \sin^2 \phi_1)}\, d\phi_1 - \frac{4z^2}{r_2} \int_0^{\frac{1}{2}\pi} \frac{d\phi_1}{\sqrt{(1 - k'^2 \sin^2 \phi_1)}}$$

or

$$I_1 = 4\left(r_2 \mathrm{E}' - \frac{z^2}{r_2} \mathrm{K}'\right), \quad \ldots \ldots \quad (12)$$

where E′ denotes, as usual, the complete elliptic integral of the second kind with modulus $k'$.

§ 3.22. The integral $I_3$ is an elliptic integral of the third kind, and the expression $a^2 + \rho^2 - 2a\rho \cos \omega'$, which occurs in it, can be replaced by $(a + \rho)^2 - k'^2 r_2^2 \sin^2 \phi_1$. If then elliptic arguments $u$ and $\alpha$ are introduced by the equations

$$\mathrm{sn}\,(u, k') = \sin \phi_1, \qquad \mathrm{sn}^2\,(\alpha, k') = \frac{r_2^2}{(a + \rho)^2}, \quad \ldots \ldots \quad (1)$$

it is found that

$$I_3 = \frac{4}{r_2 (a + \rho)^2} \int_0^{\mathrm{K}'} \frac{du}{1 - k'^2 \,\mathrm{sn}^2\,(\alpha, k')\,\mathrm{sn}^2\,(u, k')}. \quad \ldots \ldots \quad (2)$$

We consider in a general way the value of an integral of the form

$$\int_0^{\mathrm{K}} \frac{du}{1 - k^2 \,\mathrm{sn}^2 \alpha \,\mathrm{sn}^2 u} = \mathrm{J}, \text{ say}, \quad \ldots \ldots \quad (3)$$

formed with any real modulus $k$, such that $1 > k > 0$, the given value of $\mathrm{sn}^2 \alpha$ being real and greater than 1. We have at once

$$\mathrm{J} = \mathrm{K} + \int_0^{\mathrm{K}} \frac{k^2 \,\mathrm{sn}^2 \alpha \,\mathrm{sn}^2 u \,du}{1 - k^2 \,\mathrm{sn}^2 \alpha \,\mathrm{sn}^2 u} = \mathrm{K} + \frac{\mathrm{sn}\,\alpha}{\mathrm{cn}\,\alpha \,\mathrm{dn}\,\alpha} \Pi\,(\mathrm{K}, \alpha), \quad \ldots \ldots \quad (4)$$

in JACOBI's notation. Also, when $u = \mathrm{K}$, JACOBI's function $\Pi\,(u, \alpha)$ reduces to $\mathrm{KZ}\,(\alpha)$, where Z is the symbol for JACOBI's zeta function. Hence

$$\mathrm{J} = \mathrm{K}\left\{1 + \frac{\mathrm{sn}\,\alpha}{\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha} \mathrm{Z}\,(\alpha)\right\} \quad \ldots \ldots \quad (5)$$

The value of this expression is the same whatever value of $\alpha$ is chosen among those which give the right value (greater than 1) to $\mathrm{sn}^2 \alpha$, so we may assume that $\alpha$ is of the form

$$\alpha = \mathrm{K} + iv, \quad \ldots \ldots \quad (6)$$

where

$$\mathrm{K}' > v > 0. \quad \ldots \ldots \quad (7)$$

Then

$$Z(K + iv) = Z(iv) - k^2 \operatorname{sn}(iv) \operatorname{sn}(K + iv),$$

$$Z(iv) = -i\left\{Z(v, k') + \frac{\pi v}{2KK'}\right\} + i\frac{\operatorname{sn}(v, k') \operatorname{dn}(v, k')}{\operatorname{cn}(v, k')},$$

$$\operatorname{sn}(iv) = i\frac{\operatorname{sn}(v, k')}{\operatorname{cn}(v, k')},$$

$$\operatorname{sn}(K + iv) = \frac{\operatorname{cn}(iv)}{\operatorname{dn}(iv)} = \frac{1}{\operatorname{dn}(v, k')},$$

$$\frac{\operatorname{sn}\alpha}{\operatorname{cn}\alpha \operatorname{dn}\alpha} = -\frac{\operatorname{cn}(iv)\operatorname{dn}(iv)}{k'^2 \operatorname{sn}(iv)} = \frac{i}{k'^2}\frac{\operatorname{dn}(v, k')}{\operatorname{sn}(v, k')\operatorname{cn}(v, k')}.$$

From these formulæ it is found that

$$Z(\alpha) = -i\left\{Z(v, k') + \frac{\pi v}{2KK'}\right\} + ik'^2\frac{\operatorname{sn}(v, k')\operatorname{cn}(v, k')}{\operatorname{dn}(v, k')}$$

and thence that

$$J = \frac{\operatorname{dn}(v, k')}{k'^2 \operatorname{sn}(v, k')\operatorname{cn}(v, k')}\left\{KZ(v, k') + \frac{\pi v}{2K'}\right\}. \quad \text{(8)}$$

We are going to interchange $k$ and $k'$ and substitute in (2). Then the elliptic and zeta functions which occur will have modulus $k$, or $r_1/r_2$, and the elliptic argument $v$ will be such that $K > v > 0$, and that

$$\operatorname{dn}(v, k) = \frac{1}{\operatorname{sn}(\alpha, k')} = \frac{a + \rho}{r_2}. \quad \text{(9)}$$

From these conditions it follows that, with $k = r_1/r_2$,

$$\operatorname{sn}(v, k) = \frac{z}{r_1}, \qquad \operatorname{cn}(v, k) = \frac{|a - \rho|}{r_1}. \quad \text{(10)}$$

The result of the substitution is

$$I_3 = \frac{4}{z\,|a^2 - \rho^2|}\left\{K'Z(v) + \frac{\pi v}{2K}\right\}. \quad \text{(11)}$$

§ 3.23. Instead of $Z(v)$ in § 3.22 (11) it is convenient to use the notation of elliptic integrals, writing

$$Z(v) = E(v) - \frac{E}{K}v,$$

where

$$E(v) = \int_0^v \operatorname{dn}^2 u\, du, \quad E = \int_0^K \operatorname{dn}^2 u\, du,$$

and it is also convenient to replace $\pi/2K$ by the equivalent $EK'/K + E' - K'$. Then § 3.22 (11) becomes

$$I_3 = \frac{4}{z\,|a^2 - \rho^2|}\{K'\,E(v) - (K' - E')\,v\}. \qquad (1)$$

Further it is convenient to introduce a real angle $\theta'$, between 0 and $\frac{1}{2}\pi$, by the equation

$$\sin\theta' = \mathrm{sn}\,v = \frac{z}{r_1}. \qquad (2)$$

Then (1) becomes

$$I_3 = \frac{4}{z\,|a^2 - \rho^2|}\{K'E(k, \theta') - (K' - E')\,F(k, \theta')\}, \qquad (3)$$

where $E(k, \theta')$ and $F(k, \theta')$ denote the elliptic integrals

$$E(k, \theta') = \int_0^{\theta'} \sqrt{(1 - k^2\sin^2\theta')}\,d\theta', \quad F(k, \theta') = \int_0^{\theta'} \frac{d\theta'}{\sqrt{(1 - k^2\sin^2\theta')}}. \qquad (4)$$

The angle $\theta'$ can be interpreted geometrically. In fig. 4, when $\rho < a$, $\theta'$ is $\phi$, the angle $M_2M_1P$; when $\rho > a$, $\theta'$ is $\theta$, the angle $xM_1P$. It is seen that, when $\rho < a$,

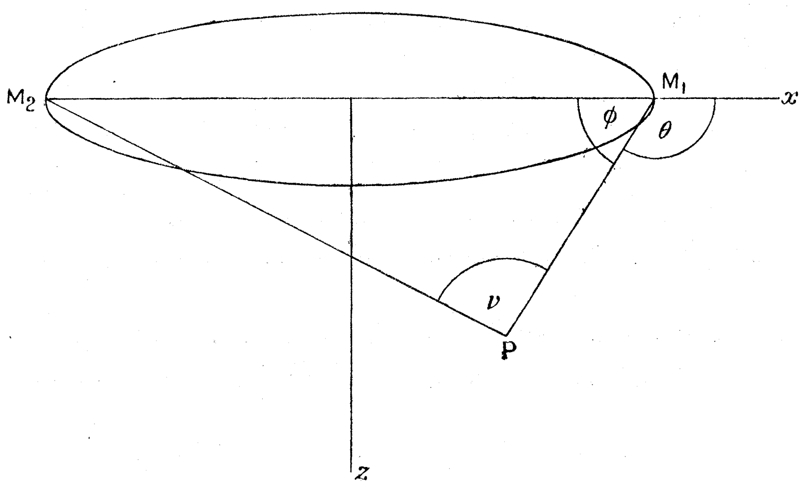


Fig. 4.

$$I_3 = \frac{4}{z\,(a^2 - \rho^2)}\{K'E(k, \phi) - (K' - E')\,F(k, \phi)\}, \qquad (5)$$

which can be shown by the use of ordinary properties of elliptic integrals to be the same as

$$I_3 = \frac{4}{z\,(a^2 - \rho^2)}[\pi - \{K'E(k, \theta) - (K' - E')\,F(k, \theta)\}]. \qquad (6)$$

Also, when $\rho > a$,

$$I_3 = \frac{4}{z(\rho^2 - a^2)} \{K' E(k, \theta) - (K' - E') F(k, \theta)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

§ 3.24. We can now return to § 3.2 (7) and evaluate $\partial\chi/\partial\rho$. Clearly we must distinguish the two cases $\rho < a$ and $\rho > a$.

When $\rho < a$ we have

$$\frac{1}{p}\frac{\partial\chi}{\partial\rho} = a^2\rho\left[\frac{\pi}{a^2} - z\left\{-\frac{1}{4a^2\rho^2}\, 4\left(r_2 E' - \frac{z^2}{r_2} K'\right) + \frac{a^2+\rho^2}{2a^2\rho^2}\frac{4}{r_2}K'\right\}\right.$$
$$\left. + z\frac{(a^2-\rho^2)^2}{4a^2\rho^2}\frac{4}{z(a^2-\rho^2)}\{K'E(k, \phi) - (K' - E')F(k, \phi)\}\right],$$

where the form § 3.23 (5) has been taken for $I_3$. This equation can be written

$$\frac{1}{\rho}\frac{\partial\chi}{\partial\rho} = p\left[\pi + \frac{zr_2}{\rho^2}E' - \frac{z(2a^2+2\rho^2+z^2)}{r_2\rho^2}K'\right.$$
$$\left. + \frac{a^2-\rho^2}{\rho^2}\{K' E(k, \phi) - (K' - E') F(k, \phi)\}\right], \quad . \quad . \quad . \quad (1)$$

in which $\frac{1}{2}\pi > \phi > 0$.

If the form § 3.23 (6) is taken for $I_3$ the formula becomes

$$\frac{1}{\rho}\frac{\partial\chi}{\partial\rho} = p\left[\pi\frac{a^2}{\rho^2} + \frac{zr_2}{\rho^2}E' - \frac{z(2a^2+2\rho^2+z^2)}{r_2\rho^2}K'\right.$$
$$\left. - \frac{a^2-\rho^2}{\rho^2}\{K' E(k, \theta) - (K' - E') F(k, \theta)\}\right] \quad . \quad . \quad . \quad (2)$$

in which $\pi > \theta > \frac{1}{2}\pi$.

When $\rho > a$ the formula (2) holds, but in it $\frac{1}{2}\pi > \theta > 0$.

For purposes of arithmetical computation it is best to use (1) when $\rho < a$ and (2) when $\rho > a$.

§ 3.3. We proceed to evaluate $\partial^2\chi/\partial\rho^2$. This will be the same as the value of $\partial^2\chi/\partial x^2$ at the point $(\rho, z)$ in the plane $\omega = 0$. Now, $p$ being constant, we have

$$\frac{1}{p}\frac{\partial^2\chi}{\partial x^2} = \iint \frac{\partial}{\partial x'}\left\{\frac{x'-x}{r(z+r)}\right\} dx'\, dy',$$

the integration being taken over the pressed area. In the same way as in § 3.2 this is transformed into

$$\frac{1}{p}\frac{\partial^2\chi}{\partial x^2} = \int_0^{2\pi} \cos\omega' \frac{a\cos\omega' - \rho\cos\omega}{r'(z+r')} a\, d\omega',$$

and hence we have

$$\frac{1}{p}\frac{\partial^2\chi}{\partial\rho^2} = \int_0^{2\pi}\frac{a^2\cos^2\omega' - a\rho\cos\omega'}{a^2+\rho^2-2a\rho\cos\omega'}\left\{1 - \frac{z}{\sqrt{(a^2+\rho^2+z^2-2a\rho\cos\omega')}}\right\}d\omega'. \quad . \quad . \quad (1)$$

In this case the identity

$$a^2 \cos^2 \omega' - a\rho \cos \omega' = \frac{(a^2 + \rho^2 - 2a\rho \cos \omega')^2}{4\rho^2} - \frac{a^2 (a^2 + \rho^2 - 2a\rho \cos \omega')}{2\rho^2} + \frac{a^4 - \rho^4}{4\rho^2} \quad . \quad . \quad . \quad (2)$$

takes the place of § 3.21 (1). It is found that

$$\int_0^{2\pi} \frac{a^2 \cos^2 \omega' - a\rho \cos \omega'}{a^2 + \rho^2 - 2a\rho \cos \omega'} d\omega' = -\frac{\pi (a^2 - \rho^2)}{2\rho^2} + \frac{\pi (a^4 - \rho^4)}{2\rho^2 |a^2 - \rho^2|}, \quad . \quad . \quad . \quad (3)$$

so that it is equal to $\pi$ if $\rho < a$, and to $-\pi a^2/\rho^2$ if $\rho > a$.

In the notation of § 3.21 (6)

$$\int_0^{2\pi} \frac{a^2 \cos^2 \omega' - a\rho \cos \omega'}{a^2 + \rho^2 - 2a\rho \cos \omega'} \frac{d\omega'}{\sqrt{(a^2 + \rho^2 + z^2 - 2a\rho \cos \omega')}}$$

$$= \frac{1}{4\rho^2} \mathrm{I}_1 - \frac{a^2}{2\rho^2} \mathrm{I}_2 + \frac{a^4 - \rho^4}{4\rho^2} \mathrm{I}_3$$

$$= \frac{1}{\rho^2} \left( r_2 \mathrm{E}' - \frac{z^2}{r_2} \mathrm{K}' \right) - \frac{2a^2}{r_2 \rho^2} \mathrm{K}' + \frac{a^4 - \rho^4}{4\rho^2} \mathrm{I}_3. \quad . \quad . \quad . \quad (4)$$

Hence it is found that, when $\rho < a$,

$$\frac{\partial^2 \chi}{\partial \rho^2} = p \left[ \pi - \frac{z r_2}{\rho^2} \mathrm{E}' + \frac{z (2a^2 + z^2)}{r_2 \rho^2} \mathrm{K}' - \frac{a^2 + \rho^2}{\rho^2} \{ \mathrm{K}' \mathrm{E} (k, \phi) - (\mathrm{K}' - \mathrm{E}') \mathrm{F} (k, \phi) \} \right], \quad (5)$$

in which $\frac{1}{2}\pi > \phi > 0$. This is the same as

$$\frac{\partial^2 \chi}{\partial \rho^2} = p \left[ -\frac{\pi a^2}{\rho^2} - \frac{z r_2}{\rho^2} \mathrm{E}' + \frac{z (2a^2 + z^2)}{r_2 \rho^2} \mathrm{K}' + \frac{a^2 + \rho^2}{\rho^2} \{ \mathrm{K}' \mathrm{E} (k, \theta) - (\mathrm{K}' - \mathrm{E}') \mathrm{F} (k, \theta) \} \right], \ (6)$$

in which $\pi > \theta > \frac{1}{2}\pi$.

When $\rho > a$, the formula (6) holds, but in it $\frac{1}{2}\pi > \theta > 0$.

§ 3.4. When there is symmetry about the axis of $z$ the equation § 1.5 (2) takes the form

$$\frac{\partial^2 \chi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \chi}{\partial \rho} + \frac{\partial \mathrm{V}}{\partial z} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and therefore, in the case of the circle under uniform pressure, we have

$$\frac{\partial \mathrm{V}}{\partial z} = -p \times 2 \{ \mathrm{K}' \mathrm{E} (k, \theta) - (\mathrm{K}' - \mathrm{E}') \mathrm{F} (k, \theta) - \mathrm{K}' k \sin \theta \}, \quad . \quad . \quad . \quad (2)$$

where $k \sin \theta$ has been written for $z/r_2$. This result, combined with § 1.5 (4), gives a formula for the solid angle $\Omega$ subtended by the circle at a point P in the form

$$\Omega = 2 \{ \mathrm{K}' \mathrm{E} (k, \theta) - (\mathrm{K}' - \mathrm{E}') \mathrm{F} (k, \theta) - \mathrm{K}' k \sin \theta \}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This formula can also be obtained in other ways. The most obvious way is to observe that, V being the potential of a thin uniform circular disc, an expression for V as an elliptic integral is known. The potential of a thin uniform elliptic disc was expressed as a single integral by F. W. DYSON.* In the case of a circular disc his formula takes the form

$$V = 2pa^2 \int_{\lambda_1}^{\infty} \sqrt{\left(1 - \frac{\rho^2}{a^2 + \psi} - \frac{z^2}{\psi}\right)} \frac{d\psi}{(a^2 + \psi)\sqrt{\psi}} \quad \ldots \ldots \quad (4)$$

where $\lambda_1$ is the positive root of the equation (for $\lambda$)

$$\frac{\rho^2}{a^2 + \lambda} + \frac{z^2}{\lambda} = 1. \quad \ldots \ldots \ldots \ldots \ldots \quad (5)$$

We have therefore

$$-\frac{1}{p}\frac{\partial V}{\partial z} = 2a^2 z \int_{\lambda_1}^{\infty} \frac{d\psi}{\psi\sqrt{\{(\psi - \lambda_1)(\psi - \lambda_2)(\psi + a^2)\}}}, \quad \ldots \ldots \quad (6)$$

where $\lambda_2$ is the negative root of the equation (5) for $\lambda$. The reduction of the elliptic integral in (6) and its expression by (3) is an exercise similar to that in §§ 3.22 and 3.23, but rather more elaborate.

Another way of obtaining the formula (3) is to observe that $\Omega$ is the velocity potential of a circular vortex, or the magnetic potential of a circular current. The motion due to the vortex can also be expressed by the Stokes stream-function $\psi$, and the field due to the current can also be expressed by the vector potential A, and $\psi$ is $A\rho$. The value of A is

$$\int_0^{2\pi} \frac{a \cos \omega' \, d\omega'}{\sqrt{(a^2 + \rho^2 + z^2 - 2a\rho \cos \omega')}},$$

so that $\psi$ is known. The relations between $\Omega$ and $\psi$ are

$$\frac{\partial \Omega}{\partial \rho} = \frac{1}{\rho}\frac{\partial \psi}{\partial z}, \quad \frac{\partial \Omega}{\partial z} = -\frac{1}{\rho}\frac{\partial \psi}{\partial \rho} \quad \ldots \ldots \ldots \ldots \quad (7)$$

The first derivatives of $\Omega$ being known, the value of $\Omega$ at a point, specified by values of $k$ and $\theta$, can be found by integrating $d\Omega$ along an arc of a suitable circle $k =$ constant, from a point where $\theta = 0$ to the point at which $\Omega$ is to be found.

The solid angle $\Omega$ subtended by the circle at a point can also be found directly from the definition of the solid angle as the area cut out on a sphere of unit radius, with its centre at the point, by a cone, standing on the circle, and having its vertex at the point. The area of a closed curve on a sphere of unit radius is the excess of $2\pi$ above the length of the supplemental curve. By this method H. A. SCHWARZ† determined the solid

* 'Quart. J. Math.,' vol. 25, p. 259 (1891). See also E. J. ROUTH, 'Analytical Statics,' vol. 2, 2nd edn., p. 129, Cambridge, 1902.

† 'Göttingen, Nachrichten,' p. 39 (1883), reprinted in 'Ges. math. Abhandlungen von H. A. SCHWARZ,' Bd. 2, p. 312, Berlin, 1890.

angle of the tangent cone from a point to an ellipsoid, and the formula (3) above can be identified with the form to which his formula reduces in the degenerate case where the ellipsoid becomes a circular disc.

§ 3.5. The remaining derivatives of V, required for the expression of the stress due to uniform pressure over a circle, can be found without difficulty, and the methods employed are similar to those used in §§ 3.2 and 3.3. It will be sufficient to write down the results. It is found that

$$\frac{\partial V}{\partial \rho} = -p\frac{r_2}{\rho}\{(1+k^2)K' - 2E'\}, \quad \ldots \quad (1)$$

$$\frac{\partial^2 V}{\partial \rho^2} = p\left[\frac{(\rho^2+a^2+z^2)(\rho^2-a^2-z^2)}{r_1{}^2 r_2 \rho^2}E' - \frac{2}{r_2}K' + \frac{r_2}{\rho^2}\{(1+k^2)K' - E'\}\right], \quad (2)$$

$$\frac{\partial^2 V}{\partial \rho\, \partial z} = p\frac{z}{r_2\rho}\left\{\left(1+\frac{1}{k^2}\right)E' - 2K'\right\}. \quad \ldots \quad (3)$$

From the equations (1) and (2), combined with the equation $\nabla^2 V = 0$, it is seen that

$$\frac{\partial^2 V}{\partial z^2} = p\left[\frac{2}{r_2}K' - \left\{\frac{r_2}{\rho^2} + \frac{(\rho^2+a^2+z^2)(\rho^2-a^2-z^2)}{r_1{}^2 r_2 \rho^2}\right\}E'\right]. \quad \ldots \quad (4)$$

The formula in the right-hand member of (4) can be simplified by using § 3.21 (7). It will be found that the coefficient of E′ becomes

$$-\frac{r_1{}^2 + r_2{}^2 - 4a^2}{r_1{}^2 r_2},$$

and, on introducing the angle $v$, or $M_1PM_2$, of fig. 4 in § 3.23, this is seen to be $-(2/r_1)\cos v$. Hence we have the result

$$z\frac{\partial^2 V}{\partial z^2} = p \times 2\sin\theta\,(K'k - E'\cos v). \quad \ldots \quad (5)$$

§ 3.6. All the relevant derivatives of $\chi$ and V have now been expressed in terms of quantities, which have been defined geometrically, and may have to be computed. We show how all these quantities may be expressed in terms of $k$ and $\theta$.

A pair of values of $k$ and $\theta$, lying in the intervals $0 < k < 1$ and $0 < \theta < \pi$, specifies a point P, so that $k$ and $\theta$ are curvilinear, but not orthogonal, co-ordinates of P. A locus $k =$ constant is a circle, which is a member of a family of coaxal circles having $M_1$ and $M_2$ as limiting points. See fig. 4 in § 3.23. A locus $\theta =$ constant is a straight line issuing from $M_1$.

When $k$ and $\theta$ are known, $\rho$, $z$, $r_1$, $r_2$, $v$ can be found.

From the figure, and the relation $r_1 = kr_2$, it is seen that

$$\rho - a = r_1\cos\theta, \quad \rho + a = r_2\Delta\theta, \quad \ldots \quad (1)$$

where

$$\Delta\theta = |\sqrt{(1-k^2\sin^2\theta)}|. \quad \ldots \quad (2)$$

Hence, to determine $r_2$, we have the equation

$$r_2 = \frac{2a}{\Delta\theta - k\cos\theta}. \quad \ldots \quad (3)$$

When $r_2$ is found from this equation, $r_1$, $\rho$, $z$ are found from the equations

$$r_1 = kr_2, \quad \rho = a + r_1 \cos\theta, \quad z = r_1 \sin\theta. \qquad (4)$$

For $v$ it is convenient to use $\phi$, or $\pi - \theta$, as a co-ordinate instead of $\theta$, and then $v$ is to be found from the equation

$$\sin v = \frac{2a}{r_2}\sin\phi, \qquad (5)$$

which is obvious from the figure, and the condition

$$v \gtreqless \tfrac{1}{2}\pi \text{ according as } \cot\phi \gtreqless k, \qquad (6)$$

which follows from the observation that $\cot\phi = k$ at any point on the semicircle having $M_1 M_2$ as diameter, and lying below $M_1 M_2$.

§ 3.7. The limiting values of the stress-components as P approaches $M_1$, the angle $\theta$ remaining constant, are found from the formulæ already given by making $k$ tend to zero. It will be sufficient to write down the limiting values of the relevant derivatives. These are given by the formulæ

$$\left.\begin{aligned} &\frac{1}{\rho}\frac{\partial\chi}{\partial\rho} = p\pi, \quad \frac{\partial^2\chi}{\partial\rho^2} = p(2\theta - \pi), \\ &\frac{\partial V}{\partial z} = -p(2\theta), \quad \frac{z}{\rho}\frac{\partial V}{\partial\rho} = 0, \\ &z\frac{\partial^2 V}{\partial\rho^2} = p\sin 2\theta, \quad z\frac{\partial^2 V}{\partial z^2} = -p\sin 2\theta, \quad z\frac{\partial^2 V}{\partial\rho\,\partial z} = 2p\sin^2\theta \end{aligned}\right\} \qquad (1)$$

From these results, and the equations § 3.1 (5), the following formulæ are found for the stress-components :—

$$\left.\begin{aligned} \widehat{\rho\rho} &= \frac{p}{2\pi}\{(1-2\sigma)\pi - 2\theta - \sin 2\theta\}, \\ \widehat{\omega\omega} &= \frac{p}{2\pi}\{-(1-2\sigma)\pi - 4\sigma\theta\}, \\ \widehat{zz} &= \frac{p}{2\pi}(-2\theta + \sin 2\theta), \\ \widehat{\rho z} &= -\frac{p}{\pi}\sin^2\theta, \end{aligned}\right\}, \qquad (2)$$

in which

$$\sigma = \frac{\lambda}{2(\lambda+\mu)},$$

or $\sigma$ is the Poisson's ratio of the material.

Very similar results would be found for a point close to the middle of one of the long sides of a long rectangle. The differences between the two cases are that, in the case of the rectangle, (i) there is nothing answering to $\widehat{\omega\omega}$, (ii) in the expression for $\widehat{xx}$, which answers to $\widehat{\rho\rho}$, there is no term answering to the term $\frac{1}{2}p(1-2\sigma)$. To elucidate this term it is appropriate to determine $\widehat{xx}$ for a point close to the middle point $M_1$ of the side DA of a rectangle when $a$ is not assumed to be small compared with $b$. See fig. 1 in § 2.1. At such a point the formulæ of §§ 2.2–2.5 show that

$$\frac{\partial V}{\partial z} = -p(2\theta_1), \quad \frac{\partial^2 \chi}{\partial x^2} = p\left(-\pi + 2\tan^{-1}\frac{b}{2a} + 2\theta_1\right), \quad z\frac{\partial^2 V}{\partial x^2} = p\sin 2\theta_1, \quad . \quad (3)$$

where $\theta_1$ is the angle so denoted in § 2.6, so that it is the analogue of $\theta$. Hence it follows that, at such a point,

$$\widehat{xx} = \frac{p}{2\pi}\left\{(1-2\sigma)\left(\pi - 2\tan^{-1}\frac{b}{2a}\right) - 2\theta_1 - \sin 2\theta_1\right\}. \quad . \quad . \quad . \quad . \quad (4)$$

When $b/a \to \infty$ the formula (4) reduces to that for a long rectangle, but, if $a/b \to \infty$ it reduces to the form for $\widehat{\rho\rho}$ in (2).

The fact that, as P approaches $M_1$, the limits of the stress-components depend upon $\theta$, which defines the direction of approach, is in accord with the theorem of the indeterminacy of stress at the bounding curve of the pressed area (§ 1.4), the pressure on the bounding plane of the solid changing discontinuously from the constant value $p$ to zero in crossing this curve. A particular case of this indeterminacy is that the stress-component $\widehat{\rho z} = 0$ if $\theta = 0$ or $\pi$, but $\widehat{\rho z} = -p/\pi$ if $\theta = \frac{1}{2}\pi$. That is to say, if first $z$ is made to tend to zero, and afterwards $\rho$ is made to tend to $a$, the value 0 is found; but, if first $\rho$ is put equal to $a$, and afterwards $z$ is made to tend to zero, the value $-p/\pi$ is found. This point was noticed by BOUSSINESQ.*

§ 3.8. We shall postpone further discussion of the stress produced by uniform pressure over an area bounded by a circle, and observe here that results already stated or obtained are sufficient for finding the displacement. For V is known from § 3.4 (4) and all the necessary derivatives have been evaluated. The results as regards the displacement of a point on the initially plane bounding surface seem to be worth recording.

For the tangential displacement $u_\rho$ it is simplest to go back to the definition of $\chi$ in § 1.1 (2). We have

$$\frac{1}{p}\left(\frac{\partial \chi}{\partial \rho}\right)_{z=0} = \iint \frac{\partial (\log r)}{\partial \rho}\, dx'\, dy', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the integration is taken over the area within the circle $\rho' = a$, and $r^2 = (x-x')^2 + (y-y')^2$. By a well-known theorem in the Theory of Attractions, this integral has the value $\pi\rho$ when $\rho < a$, and the value $\pi a^2/\rho$ when $\rho > a$. It is thus found that

$$\left.\begin{aligned} (u_\rho)_{z=0,\rho<a} &= -\frac{p}{4(\lambda+\mu)}\rho, \\ (u_\rho)_{z=0,\rho>a} &= -\frac{p}{4(\lambda+\mu)}\frac{a^2}{\rho} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These results were obtained otherwise by K. TERAZAWA (*op. cit. ante*).

* *Op cit. ante*, pp. 148, 149. *Cf.* TODHUNTER and PEARSON, *op. cit. ante*, p. 255.

For the vertical displacement $u_z$ it is simplest to evaluate V, as given by § 3.4 (4) for $z = 0$. It is found without difficulty that

$$\left.\begin{aligned}(\mathrm{V})_{z=0,\rho<a} &= 4pa\mathrm{E}, \quad \left(k = \frac{\rho}{a}\right), \\ (\mathrm{V})_{z=0,\rho>a} &= 4pa\,\frac{\mathrm{E} - \mathrm{K}k'^2}{k}, \quad \left(k = \frac{a}{\rho}\right)\end{aligned}\right\}, \qquad (3)$$

E and K having their usual meanings as complete elliptic integrals. We have then

$$(u_z)_{z=0} = \frac{\lambda + 2\mu}{4\pi\mu(\lambda+\mu)}(\mathrm{V})_{z=0}. \qquad (4)$$

The shape of the meridian curve* of the deformed bounding surface is effectively the graph of $(\mathrm{V})_{z=0}$ as a function of $\rho$. The curve consists of two arcs meeting at a junction. The arc for which $\rho < a$ looks something like a quadrant of an ellipse, concave upwards, and with minor axis vertical, but it is flatter than an ellipse towards the end of the minor axis, where $\rho = 0$, and is curved up more sharply towards the end of the major axis, where $\rho = a$. At this point, the junction, the tangent is vertical, and the curvature infinite. The ordinates of the curve at the vertex ($\rho = 0$) and the junction ($\rho = a$) are in the ratio $\pi : 2$. The other arc starts with a vertical tangent at the junction, where its curvature also is infinite, is concave downwards, and has a horizontal asymptote, coinciding with the trace of the axial plane, in which the curve lies, on the undeformed horizontal bounding plane of the solid. The curvature changes sign in passing through the infinity at the junction, so that the junction presents the appearance of a point of inflexion. The infinite curvature does not, as has been supposed, imply anything extreme in the way of extension or contraction of the radial filaments at the bounding curve of the pressed area. It appears, in fact, from (2) that all such filaments are contracted uniformly within the pressed area, the amount of the contraction being $\frac{1}{4}p/(\lambda+\mu)$, and extended outside this area, the amount of the extension being $\frac{1}{4}pa^2/(\lambda+\mu)\rho^2$. The discontinuity of strain implied in this statement is in accord with the indeterminacy of stress already noted.

### 4. *Variable Pressure over Rectangle.*

§ 4.1. In § 2.5 it was found that, if the pressure were uniform over the rectangle, the derivative $\partial^2\chi/\partial x\,\partial y$ would be infinite at the corners. Let the point $(x, y, z)$ approach the corner A for example. The stress-component $\widehat{xy}$ tends to become positively infinite of the order

$$\frac{\mu p}{2\pi(\lambda+\mu)}\log\frac{2b}{z+a_1},$$

* A scale drawing is given by K. Terazawa, *op. cit. ante*. The formulæ are effectively due to Boussinesq.

the other stress-components remaining finite. The principal stresses near the corner reduce effectively to a tension, tending to become infinite of this order, in the direction bisecting the angle DAB internally, and a numerically equal pressure in the direction bisecting this angle externally. If then the pressure over the area bounded by the rectangle were uniform, the solid could not be in equilibrium in a state of elastic strain; if the solid is in equilibrium in a state of elastic strain, under pressure applied over this area, the pressure cannot be uniform.

It was originally suggested by BOUSSINESQ (*op. cit. ante*, p. 148) that the indeterminacy of stress at the boundary of the pressed area, known to him by the special example noted at the end of § 3.7, could be evaded by supposing that the pressure is not strictly uniform over the pressed area, but tends rapidly, albeit continuously, to zero at the boundary of that area; and he stated that, if the region of continuous rapid variation were assumed to be a sufficiently narrow border, the stress at any point, not too close to the border, would be practically unaffected by the existence of the border. This expectation is verified by the result obtained in § 1.4 to the effect that, if $p = 0$ at the bounding curve of the pressed area, there is no indeterminacy of stress. If the disturbing infinity noted above had not presented itself, it would have been possible to be satisfied with this statement and verification; but its occurrence renders it desirable to attempt a closer examination of the effect of a region, bordering the region subjected to uniform pressure, and itself subjected to variable pressure, which is equal to the uniform pressure at the inner boundary of the border, and to zero at its outer boundary.

§ 4.2. The simplest case of a pressed area with a border that seems to admit of exact

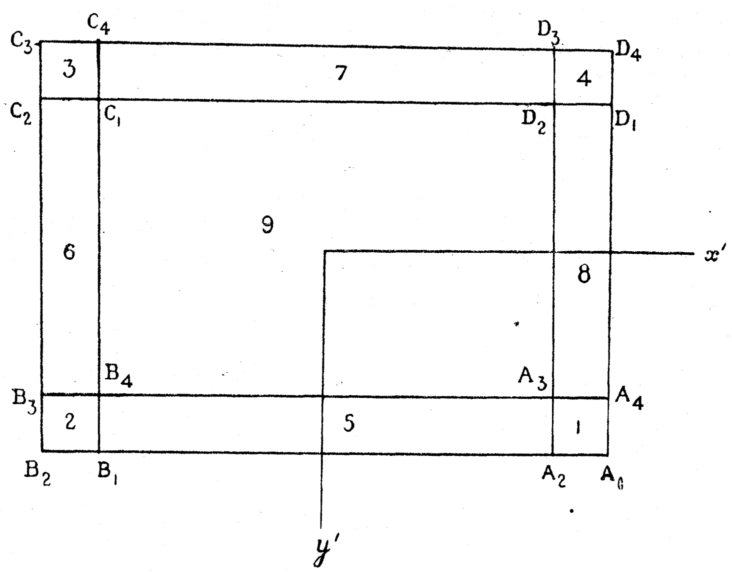


FIG. 5.

discussion is a rectangle, and the simplest law of variable pressure, constant in the bordered region, and tending to zero at the outer boundary is a bilinear law.

In fig. 5 the rectangle $A_1B_2C_3D_4$ is divided into nine regions by lines parallel to the sides and distant $c$ from them. The exact specification of the regions and of the assumed bilinear law of pressure is shown in the following table.

TABLE.

| Region. | Interval for $x'$. | Interval for $y'$. | Formula for $p$. |
|---|---|---|---|
| 1 | $a-c \ldots a$ | $b-c \ldots b$ | $p_0(a-x')(b-y')/c^2$ |
| 2 | $-a \ldots -a+c$ | $b-c \ldots b$ | $p_0(a+x')(b-y')/c^2$ |
| 3 | $-a \ldots -a+c$ | $-b \ldots -b+c$ | $p_0(a+x')(b+y')/c^2$ |
| 4 | $a-c \ldots a$ | $-b \ldots -b+c$ | $p_0(a-x')(b+y')/c^2$ |
| 5 | $-a+c \ldots a-c$ | $b-c \ldots b$ | $p_0(b-y')/c$ |
| 6 | $-a \ldots -a+c$ | $-b+c \ldots b-c$ | $p_0(a+x')/c$ |
| 7 | $-a+c \ldots a-c$ | $-b \ldots -b+c$ | $p_0(b+y')/c$ |
| 8 | $a-c \ldots a$ | $-b+c \ldots b-c$ | $p_0(a-x')/c$ |
| 9 | $-a+c \ldots a-c$ | $-b+c \ldots b-c$ | $p_0$ |

In this assumed distribution $p_0$ is constant, and $p$ is everywhere continuous.

It will be convenient to denote the distances of a point $(x, y, z)$ from the corners $A_1$, $A_2, \ldots, D_4$ of the corner squares by $a_1, a_2, \ldots, d_4$, its distance from a point on the line of junction of the regions numbered $m$ and $n$ by $r_{m,n}$, and its distance from a point on a line separating the region numbered $n$ from the external region by $r_{n,0}$. There is never any difficulty in deciding whether $r_{1,0}$, for example, refers to a point on $A_1A_4$ or on $A_1A_2$.

§ 4.3. We shall confine our attention to the evaluation of $\partial^2\chi/\partial x\, \partial y$.

The function $\chi$ is the sum of nine contributions from the nine regions numbered 1 . . . 9. We shall denote them by $\chi_1, \chi_2, \ldots, \chi_9$. We know $\partial^2\chi_9/\partial x\, \partial y$ already from § 2.5 in the form

$$\frac{\partial^2\chi_9}{\partial x\, \partial y} = p_0 \log \frac{(z+a_3)(z+c_1)}{(z+b_4)(z+d_2)}, \quad \ldots \ldots \quad (1)$$

and this can be regarded as the sum of four terms (with proper signs) such as $p_0 \log (z+a_3)$ contributed by the four corners $A_3$, $B_4$, $C_1$, $D_2$.

We consider $\partial^2\chi_1/\partial x\, \partial y$. This is given by the equation

$$\frac{\partial^2\chi_1}{\partial x\, \partial y} = \frac{p_0}{c^2}\int_{b-c}^{b}\int_{a-c}^{a} (a-x')(b-y')\, \frac{\partial^2 \{\log (z+r)\}}{\partial x'\, \partial y'}\, dx'\, dy', \quad \ldots \quad (2)$$

or, as it may be written,

$$\frac{\partial^2\chi_1}{\partial x\, \partial y} = \frac{p_0}{c^2}\int_{b-c}^{b}\int_{a-c}^{a} \left[\frac{\partial^2}{\partial x'\, \partial y'}\{(a-x')(b-y')\log (z+r)\} + \frac{\partial}{\partial x'}\{(a-x')\log (z+r)\} + \frac{\partial}{\partial y'}\{(b-y')\log (z+r)\} + \log (z+r)\right] dx'\, dy'. \quad \ldots \quad (3)$$

In this

$$\int_{b-c}^{b}\int_{a-c}^{a} \frac{\partial^2}{\partial x'\, \partial y'}\{(a-x')(b-y')\log (z+r)\}\, dx'\, dy' = c^2 \log (z+a_3), \quad \ldots \quad (4)$$

also

$$\int_{b-c}^{b}\int_{a-c}^{a}\frac{\partial}{\partial x'}\{(a-x')\log(z+r)\}\,dx'\,dy'=\int_{b-c}^{b}-c\log(z+r_{1,5})\,dy', \quad . \quad . \quad . \quad (5)$$

and

$$\int_{b-c}^{b}\int_{a-c}^{a}\frac{\partial}{\partial y'}\{(b-y')\log(z+r)\}\,dx'\,dy'=\int_{a-c}^{a}-c\log(z+r_{1,8})\,dx'. \quad . \quad . \quad (6)$$

We next consider $\partial^2\chi_5/\partial x\,\partial y$. This is given by the equation

$$\frac{\partial^2\chi_5}{\partial x\,\partial y}=\frac{p_0}{c}\int_{b-c}^{b}\int_{-a+c}^{a-c}(b-y')\frac{\partial^2\{\log(z+r)\}}{\partial x'\,\partial y'}\,dx'\,dy', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is

$$\frac{\partial^2\chi_5}{\partial x\,\partial y}=\frac{p_0}{c}\int_{b-c}^{b}(b-y')\frac{\partial}{\partial y'}\{\log(z+r_{1,5})-\log(z+r_{2,5})\}\,dy'. \quad . \quad . \quad . \quad . \quad (8)$$

Now

$$\begin{aligned}\int_{b-c}^{b}(b-y')\frac{\partial}{\partial y'}\{\log(z+r_{1,5})\}\,dy' \\ &=\int_{b-c}^{b}\left[\frac{\partial}{\partial y'}\{(b-y')\log(z+r_{1,5})\}+\log(z+r_{1,5})\right]dy' \\ &=-c\log(z+a_3)+\int_{b-c}^{b}\log(z+r_{1,5})\,dy'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)\end{aligned}$$

The first term yields to $\partial^2\chi/\partial x\,\partial y$ a contribution $-p_0\log(z+a_3)$, and the second term yields a contribution which cancels that yielded by (5).

The second term of (8) can be combined in the same way with the contributions of the corner square $B_1B_2B_3B_4$.

In the same way $\partial^2\chi_8/\partial x\,\partial y$ will be the sum of two integrals. One of them yields to $\partial^2\chi/\partial x\,\partial y$ a contribution $-p_0\log(z+a_3)$, and a contribution which cancels that yielded by (6). The other can be combined as above with the contributions of the corner square $D_1D_2D_3D_4$.

In this way all the contributions of all the regions (including 9) can be arranged as contributions associated with the four corner squares. The contribution associated with the corner square $A_1A_2A_3A_4$ is

$$\frac{p_0}{c^2}\int_{b-c}^{b}\int_{a-c}^{a}\log(z+r)\,dx'\,dy'=\mathrm{F}(x,y,z),\text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

To see how the process works out for any other corner square we take, for example, $B_1B_2B_3B_4$. We have

$$\begin{aligned}\frac{\partial^2\chi_2}{\partial x\,\partial y}&=\frac{p_0}{c^2}\int_{b-c}^{b}\int_{-a}^{-a+c}(a+x')(b-y')\frac{\partial^2\{\log(z+r)\}}{\partial x'\,\partial y'}\,dx'\,dy' \\ &=\frac{p_0}{c^2}\int_{b-c}^{b}\int_{-a}^{-a+c}\left[\frac{\partial^2}{\partial x'\,\partial y'}\{(a+x')(b-y')\log(z+r)\}+\frac{\partial}{\partial x'}\{(a+x')\log(z+r)\}\right. \\ &\qquad\left.-\frac{\partial}{\partial y'}\{(b-y')\log(z+r)\}-\log(z+r)\right]dx'\,dy'.\end{aligned}$$

All the contributions associated with this corner square reduce to

$$-\frac{p_0}{c^2}\int_{b-c}^{b}\int_{-a}^{-a+c}\log(z+r)\,dx'\,dy'.$$

If in this we put $x' = -x''$ we shall get $(x+x'')$ instead of $(x-x')$ in the expression for $r$, and a form

$$-\frac{p_0}{c^2}\int_{b-c}^{b}\int_{a-c}^{a}\log(z+r)\,dx''\,dy',$$

where $r^2 = (x''+x)^2+(y'-y)^2+z^2$, and $r$ is positive. This is $-\mathrm{F}(-x, y, z)$.

By proceeding in this way it is seen that the final result is

$$\frac{\partial^2\chi}{\partial x\,\partial y} = \mathrm{F}(x, y, z) - \mathrm{F}(-x, y, z) + \mathrm{F}(-x, -y, z) - \mathrm{F}(x, -y, z), \quad . \quad . \quad (11)$$

where $\mathrm{F}(x, y, z)$ is given by (10).

§ 4.4. We have to evaluate the integral in § 4.3 (10). We have at once

$$\int_{a-c}^{a}\log(z+r)\,dx' = (a-x)\log(z+r_{1,0}) - (a-c-x)\log(z+r_{1,5}) - \int_{a-c}^{a}\frac{(x'-x)^2}{r(z+r)}\,dx', \quad . \quad . \quad . \quad (1)$$

in which $r_{1,0}$ denotes the distance of the point $(x, y, z)$ from a point on $\mathrm{A}_4\mathrm{A}_1$. The integral in the right-hand member of (1) can be evaluated by methods used in § 2.2, and hence it is found that

$$\begin{aligned}\int_{a-c}^{a}\log(z+r)\,dx' = &-c + (a-x)\log(z+r_{1,0}) - (a-c-x)\log(z+r_{1,5})\\ &- z\log(a-x+r_{1,0}) + z\log(a-c-x+r_{1,5})\\ &+ (y'-y)\left(\tan^{-1}\frac{a-x}{y'-y} - \tan^{-1}\frac{a-c-x}{y'-y}\right)\\ &- (y'-y)\left\{\tan^{-1}\frac{z(a-x)}{(y'-y)r_{1,0}} - \tan^{-1}\frac{z(a-c-x)}{(y'-y)\,r_{1,5}}\right\} \quad . \quad . \quad (2)\end{aligned}$$

To evaluate the integral in § 4.3 (10), we take the terms in the right-hand member of (2) separately, omitting the first because its integral is obvious, the third because its integral can be deduced from that of the second, and so on. We have

$$\begin{aligned}\int_{b-c}^{b}\log(z+r_{1,0})\,dy' = &-c + (b-y)\log(z+a_1) - (b-c-y)\log(z+a_4)\\ &+ z\log(b-y+a_1) - z\log(b-c-y+a_4)\\ &+ (a-x)\left(\tan^{-1}\frac{b-y}{a-x} - \tan^{-1}\frac{b-c-y}{a-x}\right)\\ &- (a-x)\left\{\tan^{-1}\frac{z(b-y)}{(a-x)\,a_1} - \tan^{-1}\frac{z(b-c-y)}{(a-x)\,a_4}\right. . \quad (3)\end{aligned}$$

Also

$$\int_{b-c}^{b} \log (a - x + r_{1,0})\, dy' = -c + (b-y) \log (a-x+a_1) - (b-c-y) \log (a-x+a_4)$$
$$+ (a-x) \log (b-y+a_1) - (a-x) \log (b-c-y+a_4)$$
$$+ z \left(\tan^{-1} \frac{b-y}{z} - \tan^{-1} \frac{b-c-y}{z}\right)$$
$$-z \left\{\tan^{-1} \frac{(b-y)(a-x)}{za_1} - \tan^{-1} \frac{(b-c-y)(a-x)}{za_4}\right\}. \quad (4)$$

Further

$$\int_{b-c}^{b} (y' - y) \tan^{-1} \frac{a-x}{y'-y}\, dy' = \tfrac{1}{2} (a-x)\, c + \tfrac{1}{2} (b-y)^2 \tan^{-1} \frac{a-x}{b-y}$$
$$- \tfrac{1}{2} (b-c-y)^2 \tan^{-1} \frac{a-x}{b-c-y} - \tfrac{1}{2} (a-x)^2 \left\{\tan^{-1} \frac{b-y}{a-x} - \tan^{-1} \frac{b-c-y}{a-x}\right\}. \quad (5)$$

Finally

$$\int_{b-c}^{b} (y'-y) \tan^{-1} \frac{z(a-x)}{(y'-y)\, r_{1,0}}\, dy' = (a-x)\, z \log \frac{b-y+a_1}{b-c-y+a_4}$$
$$+ \tfrac{1}{2} (b-y)^2 \tan^{-1} \frac{z(a-x)}{(b-y)\, a_1} - \tfrac{1}{2} (b-c-y)^2 \tan^{-1} \frac{z(a-x)}{(b-c-y)\, a_4}$$
$$- \tfrac{1}{2} (a-x)^2 \left\{\tan^{-1} \frac{z(b-y)}{(a-x)\, a_1} - \tan^{-1} \frac{z(b-c-y)}{(a-x)\, a_4}\right\}$$
$$- \tfrac{1}{2} z^2 \left\{\tan^{-1} \frac{(a-x)(b-y)}{za_1} - \tan^{-1} \frac{(a-x)(b-c-y)}{za_4}\right\}. \quad . \quad . \quad (6)$$

The complete result of the integration with respect to $y'$ can now be written down, and it is seen that the function F $(x, y, z)$ of § **4.3** (10) is the sum of $-\frac{3}{2}p_0$ and four sets of terms associated with the four corners of the corner square $A_1A_2A_3A_4$, so that we may write

$$F(x, y, z) = -\tfrac{3}{2}p_0 + G(a, b, x, y, z) - G(a-c, b, x, y, z) - G(a, b-c, x, y, z)$$
$$+ G(a-c, b-c, x, y, z), \quad . \quad . \quad . \quad . \quad (7)$$

where

$$G(a, b, x, y, z) = \frac{p_0}{c^2} \Big[ (a-x)(b-y) \log (z+a_1) - z(a-x) \log (b-y+a_1)$$
$$- z(b-y) \log (a-x+a_1)$$
$$+ \tfrac{1}{2} (a-x)^2 \left\{\tan^{-1} \frac{b-y}{a-x} - \tan^{-1} \frac{z(b-y)}{(a-x)\, a_1}\right\}$$
$$+ \tfrac{1}{2} (b-y)^2 \left\{\tan^{-1} \frac{a-x}{b-y} - \tan^{-1} \frac{z(a-x)}{(b-y)\, a_1}\right\}$$
$$+ \tfrac{3}{2} z^2 \tan^{-1} \frac{(a-x)(b-y)}{za_1}\Big]. \quad . \quad (8)$$

As usual, all the inverse tangents have values between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. In forming G $(a-c, b, x, y, z), \ldots$, the suffix 1 in $a_1$ must be changed in a way that is clear from the observation that $a_1{}^2 = (a-x)^2 + (b-y)^2 + z^2$. With (7) and (8) as defining equations $\partial^2\chi/\partial x\, \partial y$ is given by § 4.3 (11).

§ 4.5. The result shows that the disturbing infinity pointed out in § 4.1 is removed altogether when the discontinuity of pressure at the boundary of the pressed area is replaced by a continuous transition. With the assumed distribution of pressure every logarithm, which can become infinite at a point, is multiplied by an expression, which vanishes at the point, in an order sufficiently high to secure that the product has the limit zero at the point. As $\partial^2\chi/\partial x\, \partial y$ is the only quantity, occurring in the expressions for the stress-components, which can become infinite, it follows that there is no infinite stress, and, in particular, there is no infinite tensile stress.

The inverse tangents that present themselves in such an expression as G $(a, b, x, y, z)$ are indeterminate at corners of a corner square, but never infinite, and each of them is multiplied by an expression, which vanishes at the corner where the inverse tangent becomes indeterminate. Thus the infinity is not removed at the expense of introducing a new indeterminacy.

The discussion has been restricted to the evaluation of $\partial^2\chi/\partial x\, \partial y$ for the assumed law of pressure, because this was the derivative which became infinite when the pressure was assumed to be uniform over the pressed area, and discontinuous at its boundary. It is clear that the other derivatives required to express the stress-components, for the assumed law of pressure, could be evaluated by processes similar to those used above. The results would not be of much interest. The fact that the stress at every point would be determinate is in accordance with the result of § 1.4.

It is important to observe that the breadth $c$ of the border must be finite. It would not be possible to pass to a limit by diminishing $c$ indefinitely. The only result would be to reproduce § 2.5 (1) with its disturbing infinity.

It may be noted that in the course of the work the double integral

$$\iint \log(z+r)\, dx'\, dy',$$

taken over a square, has been evaluated, and there is nothing in the processes that have been employed to restrict their application to a square. They would apply equally to any rectangle. Thus the function $\chi$ for uniform pressure over any rectangle could be found if desired. Since V is $\partial\chi/\partial z$, the value of V could be deduced, and any derivative of V or $\chi$ could be evaluated by differentiation. It is therefore possible to obtain a formula for the displacement due to uniform pressure over a rectangle, but it does not appear that any useful purpose would be served by carrying out the calculation.

## 5. *Variable Pressure over Circle.*

§ 5.1. In further illustration of methods of calculating the stress due to pressure which varies over the pressed area, we take the case where the area is a circle and the pressure is distributed symmetrically around the centre. We show first how $\partial\chi/\partial\rho$ can be found if $\partial V/\partial z$ and $\partial V/\partial\rho$ are known, and then proceed to discuss a particular case for which V is known.

Equations 1.1 (4) give

$$\left.\begin{aligned}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\chi}{\partial\rho}\right) &= -\,\rho\,\frac{\partial V}{\partial z},\\ \frac{\partial}{\partial z}\left(\rho\,\frac{\partial\chi}{\partial\rho}\right) &= \rho\,\frac{\partial V}{\partial\rho}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

It follows that, if the value of $\partial\chi/\partial\rho$ is known at one point, its value at the point $(\rho, z)$ can be expressed by the equation

$$\rho\,\frac{\partial\chi}{\partial\rho} = \rho_0\left(\frac{\partial\chi}{\partial\rho}\right)_0 + \int\left(-\,\rho\,\frac{\partial V}{\partial z}\,d\rho + \rho\,\frac{\partial V}{\partial\rho}\,dz\right), \quad \ldots\ldots \quad (2)$$

where $(\partial\chi/\partial\rho)_0$ is the known value at the starting point (where $\rho = \rho_0$), and the integral is a line integral taken along any suitable curve from that point to the point $(\rho, z)$.

Let P denote the resultant pressure on the circular pressed area, so that

$$P = \iint p\,dx'\,dy', \quad \ldots\ldots\ldots\ldots \quad (3)$$

where the integral is taken over this area. The value of $\chi$ at any point M in the plane $z = 0$ is given by the equation

$$\chi = \iint p\log r\,dx'\,dy', \quad \ldots\ldots\ldots\ldots \quad (4)$$

where the integral is taken over this area and $r^2 = (x - x')^2 + (y - y')^2$. If M is outside the circular boundary it is known that the value of $\partial\chi/\partial\rho$ at M is $P/\rho_0$, where $\rho_0$ is the value of $\rho$ at M. Hence we have

$$\rho\,\frac{\partial\chi}{\partial\rho} = P + \int\left(-\,\rho\,\frac{\partial V}{\partial z}\,d\rho + \rho\,\frac{\partial V}{\partial\rho}\,dz\right), \quad \ldots\ldots\ldots \quad (5)$$

provided that the path of integration starts at a point M, on the plane $z = 0$, and outside the boundary of the pressed area.

§ 5.2. The particular case which we are going to consider is that where $p$, as a function of $\rho'$, is given by the formula

$$p = \frac{3P}{2\pi a^2}\sqrt{\left(1 - \frac{\rho'^2}{a^2}\right)}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

$a$ being the radius of the circle. This is sometimes described as a "hemispherical" distribution of pressure.

The problem has been much discussed, being the special case for a circular area of the corresponding problem for an area bounded by an ellipse, which problem was utilised by HERTZ in his theory of the displacement produced in a body against which another presses (*op. cit. ante*). HERTZ succeeded in finding the normal displacement of the surface of either body, but failed to determine the stress. In the case of the circle the solution was completed by M. T. HUBER.* On the basis of HUBER'S formulæ the lines of stress in the meridian plane were traced by S. FUCHS.† The problem is of such interest, and affords such an excellent example of the method indicated in § 5.1, that no apology seems to be needed for proposing a new solution.

When $p$ is given by (1), V is the limiting form to which the potential of a uniform solid oblate spheroid, of minor axis $2c$ and density $\rho_1$, and having the circular boundary of the pressed area as its equator, tends, when $\rho_1$, $c$ satisfy the conditions

$$\rho_1 \to \infty, \qquad c \to 0, \qquad 2\rho_1 c = \frac{3\mathrm{P}}{2\pi a^2}. \qquad (2)$$

We can therefore write down the formula

$$\mathrm{V} = \frac{3\mathrm{P}}{4}\int_{\lambda_1}^{\infty}\left(1 - \frac{\rho^2}{a^2+\psi} - \frac{z^2}{\psi}\right)\frac{d\psi}{(a^2+\psi)\sqrt{\psi}}, \qquad (3)$$

where $\lambda_1$ is the positive root of the equation (for $\lambda$)

$$\frac{\rho^2}{a^2+\lambda} + \frac{z^2}{\lambda} = 1. \qquad (4)$$

With this form for V we have

$$\frac{\partial \mathrm{V}}{\partial \rho} = -\frac{3\mathrm{P}}{2}\,\rho\int_{\lambda_1}^{\infty}\frac{d\psi}{(a^2+\psi)^2\sqrt{\psi}}, \qquad (5)$$

and

$$\frac{\partial \mathrm{V}}{\partial z} = -\frac{3\mathrm{P}}{2}\,z\int_{\lambda_1}^{\infty}\frac{d\psi}{(a^2+\psi)\,\psi^{3/2}}. \qquad (6)$$

The integrals in (3), (5), (6) can be evaluated, and give

$$\mathrm{V} = \frac{3\mathrm{P}}{4}\left\{\frac{2}{a}\tan^{-1}\frac{a}{\sqrt{\lambda_1}} - \frac{\rho^2}{a^3}\left(\tan^{-1}\frac{a}{\sqrt{\lambda_1}} - \frac{a\sqrt{\lambda_1}}{a^2+\lambda_1}\right) - \frac{2z^2}{a^3}\left(\frac{a}{\lambda_1} - \tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right)\right\}, \qquad (7)$$

with

$$\frac{\partial \mathrm{V}}{\partial \rho} = -\frac{3\mathrm{P}}{2}\frac{\rho}{a^3}\left(\tan^{-1}\frac{a}{\sqrt{\lambda_1}} - \frac{a\sqrt{\lambda_1}}{a^2+\lambda_1}\right), \qquad (8)$$

and

$$\frac{\partial \mathrm{V}}{\partial z} = -3\mathrm{P}\frac{z}{a^3}\left(\frac{a}{\sqrt{\lambda_1}} - \tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right). \qquad (9)$$

* 'Ann. Physik,' (Folge 4), vol. 14, p. 153 (1904).

† 'Physikalische Zeitschrift,' vol. 14, p. 1282 (1913).

§ 5.3. The remaining derivatives of V that are required can be obtained from § 5.2 (5) and (6). From § 5.2 (6) we have

$$\frac{\partial^2 V}{\partial z^2} = -\frac{3P}{2}\int_{\lambda_1}^{\infty}\frac{d\psi}{(a^2+\psi)\,\psi^{3/2}} + \frac{3P}{2}\,z\,\frac{1}{(a^2+\lambda_1)\,\lambda_1^{3/2}}\,\frac{\partial\lambda_1}{\partial z}, \quad \ldots \ldots \quad (1)$$

where $\partial\lambda_1/\partial z$ is found from § 5.2 (4), with $\lambda_1$ written for $\lambda$, in the form

$$\frac{\partial\lambda_1}{\partial z} = \frac{2z\lambda_1\,(a^2+\lambda_1)^2}{\lambda_1^2\rho^2 + (a^2+\lambda_1)^2 z^2},$$

which is the same as

$$\frac{\partial\lambda_1}{\partial z} = \frac{2z\lambda_1\,(a^2+\lambda_1)}{\lambda_1^2 + a^2z^2}. \quad \ldots \ldots \quad (2)$$

We have therefore

$$\frac{\partial^2 V}{\partial z^2} = -\frac{3P}{a^3}\left(\frac{a}{\sqrt{\lambda_1}} - \tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right) + \frac{3Pz^2}{(\lambda_1^2 + a^2z^2)\sqrt{\lambda_1}}. \quad \ldots \ldots \quad (3)$$

From this equation, combined with § 5.2 (8) and the equation $\nabla^2 V = 0$, it is found that

$$\frac{\partial^2 V}{\partial \rho^2} = -\frac{3P}{2a^3}\left(\tan^{-1}\frac{a}{\sqrt{\lambda_1}} - \frac{a\sqrt{\lambda_1}}{a^2+\lambda_1}\right) + \frac{3P\,(\lambda_1 - z^2)\sqrt{\lambda_1}}{(\lambda_1^2 + a^2z^2)(a^2+\lambda_1)}. \quad \ldots \quad (4)$$

From the equation § 5.2 (5) it is found, in the manner in which (3) was obtained, that

$$\frac{\partial^2 V}{\partial\rho\,\partial z} = \frac{3P\rho z\sqrt{\lambda_1}}{(\lambda_1^2 + a^2z^2)(a^2+\lambda_1)}. \quad \ldots \ldots \quad (5)$$

§ 5.4. In order to determine $\partial\chi/\partial\rho$ from the equations §§ 5.1 (5), 5.2 (8), and 5.2 (9), the path of integration is taken to be the section of the spheroid

$$\frac{\rho^2}{a^2+\lambda_1} + \frac{z^2}{\lambda_1} = 1, \quad \ldots \ldots \quad (1)$$

by the axial plane passing through the point $(\rho, z)$ at which $\partial\chi/\partial\rho$ is to be found. In terms of the usual conjugate functions $\xi$ and $\eta$, associated with confocal ellipses, we have

$$\rho = a\cosh\xi\cos\eta, \quad z = a\sinh\xi\sin\eta, \quad \sqrt{\lambda_1} = a\sinh\xi. \quad \ldots \quad (2)$$

In the integration $\xi$ is constant, and, at the starting point, $\eta = 0$. The integral in § 5.1 (5) becomes

$$-3P\int_0^{\eta}\left\{\left(\frac{1}{\sinh\xi} - \tan^{-1}\frac{1}{\sinh\xi}\right)\cosh^2\xi\sinh\xi\sin^2\eta\cos\eta \right.$$
$$\left. + \tfrac{1}{2}\left(\tan^{-1}\frac{1}{\sinh\xi} - \frac{\sinh\xi}{\cosh^2\xi}\right)\cosh^2\xi\sinh\xi\cos^3\eta\right\}d\eta,$$

which is

$$\mathrm{P}\left\{\tan^{-1}\left(\frac{1}{\sinh\xi}\right)\cosh^2\xi\sinh\xi\,(\sin^3\eta-\tfrac{3}{2}\sin\eta+\tfrac{1}{2}\sin^3\eta)\right.$$
$$\left.-\cosh^2\xi\sin^3\eta+\sinh^2\xi\,(\tfrac{3}{2}\sin\eta-\tfrac{1}{2}\sin^3\eta)\right\}.$$

Hence, after a little reduction, it is found that

$$\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}=\frac{\mathrm{P}}{\rho^2}+\frac{3\mathrm{P}}{2}\frac{z}{a^3}\left(\frac{a}{\sqrt{\lambda_1}}-\tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right)-\frac{\mathrm{P}z}{\rho^2\sqrt{\lambda_1}}\left(\tfrac{3}{2}-\tfrac{1}{2}\frac{z^2}{\lambda_1}\right).\quad . \quad . \quad (3)$$

From this result, combined with the equations §§ 5.2 (9) and 3.4 (1) it is found that

$$\frac{\partial^2\chi}{\partial\rho^2}=-\frac{\mathrm{P}}{\rho^2}+\frac{3\mathrm{P}}{2}\frac{z}{a^3}\left(\frac{a}{\sqrt{\lambda_1}}-\tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right)+\frac{\mathrm{P}z}{\rho^2\sqrt{\lambda_1}}\left(\tfrac{3}{2}-\tfrac{1}{2}\frac{z^2}{\lambda_1}\right).\quad . \quad . \quad (4)$$

§ 5.5. The expressions for the relevant stress-components can be written down, after a little reduction, in the forms

$$\widehat{\rho\rho}=\frac{3\mathrm{P}}{2\pi a^2}\left[\frac{1-2\sigma}{3}\frac{a^2}{\rho^2}\left\{1-\left(\frac{z}{\sqrt{\lambda_1}}\right)^3\right\}+(1+\sigma)\frac{z}{a}\tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right.$$
$$\left.+(1-\sigma)\frac{z\sqrt{\lambda_1}}{a^2+\lambda_1}-2\frac{z}{\sqrt{\lambda_1}}+\frac{a^2z^3}{(\lambda_1^2+a^2z^2)\sqrt{\lambda_1}}\right],$$

$$\widehat{\omega\omega}=\frac{3\mathrm{P}}{2\pi a^2}\left[-\frac{1-2\sigma}{3}\frac{a^2}{\rho^2}\left\{1-\left(\frac{z}{\sqrt{\lambda_1}}\right)^3\right\}+(1+\sigma)\frac{z}{a}\tan^{-1}\frac{a}{\sqrt{\lambda_1}}\right.$$
$$\left.-(1-\sigma)\frac{z\sqrt{\lambda_1}}{a^2+\lambda_1}-2\sigma\frac{z}{\sqrt{\lambda_1}}\right],$$

$$\widehat{zz}=-\frac{3\mathrm{P}}{2\pi}\frac{z^3}{(\lambda_1^2+a^2z^2)\sqrt{\lambda_1}},\qquad \widehat{\rho z}=-\frac{3\mathrm{P}}{2\pi}\frac{\rho z^2\sqrt{\lambda_1}}{(\lambda_1^2+a^2z^2)(a^2+\lambda_1)}.$$

These are equivalent to the forms found by M. T. HUBER (*op. cit. ante*), except that, in his paper, the coefficient of $z\sqrt{\lambda_1}/(a^2+\lambda_1)$ in what is here called $\widehat{\omega\omega}$ is misprinted "$-(1+\mu)$" instead of "$-(1-\mu)$," his $\mu$ being my $\sigma$. The misprint does not affect the calculation of the lines of stress by S. FUCHS.

§ 5.6. To evaluate the limits to which the stress-components tend as the point $(\rho, z)$ moves to a point $\mathrm{M}_1$ on the bounding circle of the pressed area, it is convenient to take $\mathrm{M}_1$ to be the point at which both $\xi$ and $\eta$ vanish. Since

$$\sin\eta=\frac{z}{\sqrt{\lambda_1}},$$

it is seen that $z$, $\lambda_1$ and $z/\sqrt{\lambda_1}$ all vanish at $\mathrm{M}_1$. It follows that

$$\frac{a^2z^2}{\lambda_1^2+a^2z^2}$$

has the limit zero at $\mathrm{M}_1$. From these remarks it is seen that, at $\mathrm{M}_1$,

$$\widehat{\rho\rho}=(1-2\sigma)\frac{\mathrm{P}}{2\pi a^2},\quad \widehat{\omega\omega}=-(1-2\sigma)\frac{\mathrm{P}}{2\pi a^2},\quad \widehat{zz}=0,\quad \widehat{\rho z}=0.\quad . \quad . \quad . \quad (1)$$

If these formulæ are compared with those in § 3.7 (2) it appears that the terms of § 3.7 (2) which have no determinate limits at $M_1$ have disappeared, and, in the remaining terms, $P/\pi a^2$ has taken the place of $p$. It is not difficult to prove that this result does not depend upon the formula § 5.2 (1) for $p$, but would hold for any distribution of pressure, symmetrical about the centre of a pressed circular area, and vanishing at its boundary.

When $p$ is given, as a function of $\rho'$, by the formula § 5.2 (1), the derivatives $\partial p/\partial x'$ and $\partial p/\partial y'$ become infinite at the bounding curve $\rho' = a$. The proved determinacy of stress at this curve illustrates the statement that the argument of §§ 1.3 and 1.4 would not be affected if the normal derivative of $p$ at $s$ were infinite.

## 6. *Numerical Discussion.*

§ 6.1. We shall now proceed to record the results of a numerical and approximate calculation of the values of the components of stress, and the principal stresses, in the case of uniform pressure over a circular area, considered in Chapter 3 above, using the other solutions that have been obtained to illustrate some points that arise. The values recorded will be those for a selected set of points. These are situated on the circles for which $k$ has the values sin 5°, sin 15°, sin 30°, sin 50°, and on the radii, from such a point as $M_1$ in fig. 4 of § 3.23, specified by the values 0°, 10°, 20°, . . ., 180° of $\phi$. With additional labour the tabulation might be carried out for more numerous points, closer together, but the selected set appears to be sufficiently numerous, and suitably placed, to indicate the character of the stress-distribution. For the purpose of the calculation the value $\frac{1}{4}$ has been taken for $\sigma$.

Table I shows the approximate values of $\rho/a$ and $z/a$ at the selected points.

Table II shows the approximate values of $\widehat{\rho\rho}/p$ and $\widehat{zz}/p$, and Table III those of $\widehat{\rho z}/p$ and $\widehat{\omega\omega}/p$, at the selected points. The signs are omitted. All the values of $\widehat{zz}$ and $\widehat{\rho z}$ are negative. Some of the values of $\widehat{\rho\rho}$ and $\widehat{\omega\omega}$ are positive. In Table II the values below the thick horizontal lines are positive, those above them negative. In Table III the values between the thick horizontal lines are positive, the others negative.

§ 6.2. The chart of the stress-distribution is completed by taking account of the formulæ § 3.7 (2), in which $\sigma$ should be replaced by $\frac{1}{4}$, and by calculating the values of the stress-components for points on the axis of $z$. The formulæ § 3.7 (2) give the stress at points close to the circular edge of the pressed area. The stress at points on the axis of $z$ could be found by evaluating the limits to which the relevant derivatives of V and $\chi$ tend as $k \rightarrow 1$, but it is simpler to observe that

$$\left.\begin{aligned} (\mathrm{V})_{\rho=0} &= 2\pi p\,\{\sqrt{(a^2+z^2)} - z\}, \\ \lim_{\rho\to 0}\frac{\partial^2 \mathrm{V}}{\partial\rho^2} &= \lim_{\rho\to 0}\frac{1}{\rho}\frac{\partial \mathrm{V}}{\partial\rho} = -\tfrac{1}{2}\frac{\partial^2}{\partial z^2}(\mathrm{V})_{\rho=0}, \\ \lim_{\rho\to 0}\frac{\partial^2 \chi}{\partial\rho^2} &= \lim_{\rho\to 0}\frac{1}{\rho}\frac{\partial \chi}{\partial\rho} = -\tfrac{1}{2}\frac{\partial}{\partial z}(\mathrm{V})_{\rho=0} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

TABLE I.

| $\phi$. | $k = \sin 5°$. | | $k = \sin 15°$. | | $k = \sin 30°$. | | $k = \sin 50°$. | |
|---|---|---|---|---|---|---|---|---|
| | $\rho/a$. | $z/a$. | $\rho/a$. | $z/a$. | $\rho/a$. | $z/a$. | $\rho/a$. | $z/a$. |
| ° | | | | | | | | |
| 0 | 0·840 | 0 | 0·589 | 0 | 0·323 | 0 | 0·132 | 0 |
| 10 | 0·842 | 0·028 | 0·593 | 0·072 | 0·338 | 0·117 | 0·136 | 0·152 |
| 20 | 0·849 | 0·055 | 0·607 | 0·143 | 0·354 | 0·235 | 0·146 | 0·311 |
| 30 | 0·860 | 0·081 | 0·631 | 0·213 | 0·382 | 0·357 | 0·164 | 0·483 |
| 40 | 0·875 | 0·105 | 0·665 | 0·277 | 0·424 | 0·483 | 0·195 | 0·676 |
| 50 | 0·894 | 0·127 | 0·710 | 0·346 | 0·484 | 0·615 | 0·244 | 0·901 |
| 60 | 0·916 | 0·145 | 0·766 | 0·406 | 0·566 | 0·752 | 0·323 | 1·17 |
| 70 | 0·942 | 0·160 | 0·833 | 0·460 | 0·675 | 0·892 | 0·452 | 1·51 |
| 80 | 0·970 | 0·170 | 0·911 | 0·504 | 0·819 | 1·03 | 0·663 | 1·91 |
| 90 | 1 | 0·175 | 1 | 0·536 | 1 | 1·15 | 1 | 2·28 |
| 100 | 1·03 | 0·175 | 1·10 | 0·553 | 1·22 | 1·26 | 1·51 | 2·88 |
| 110 | 1·06 | 0·169 | 1·20 | 0·552 | 1·48 | 1·32 | 2·21 | 3·33 |
| 120 | 1·09 | 0·158 | 1·31 | 0·530 | 1·77 | 1·33 | 3·10 | 3·63 |
| 130 | 1·12 | 0·142 | 1·41 | 0·487 | 2·07 | 1·27 | 4·10 | 3·70 |
| 140 | 1·14 | 0·120 | 1·50 | 0·422 | 2·36 | 1·14 | 5·14 | 3·47 |
| 150 | 1·16 | 0·094 | 1·58 | 0·337 | 2·62 | 0·934 | 6·10 | 2·94 |
| 160 | 1·18 | 0·065 | 1·65 | 0·235 | 2·82 | 0·664 | 6·87 | 2·14 |
| 170 | 1·19 | 0·033 | 1·69 | 0·121 | 2·96 | 0·345 | 7·37 | 1·12 |
| 180 | 1·19 | 0 | 1·70 | 0 | 3 | 0 | 7·55 | 0 |

TABLE II.

| $\phi$. | $k = \sin 5°$. | | $k = \sin 15°$. | | $k = \sin 30°$. | | $k = \sin 50°$. | |
|---|---|---|---|---|---|---|---|---|
| | $\widehat{\rho\rho}/p$. | $\widehat{zz}/p$. | $\widehat{\rho\rho}/p$. | $\widehat{zz}/p$. | $\widehat{\rho\rho}/p$. | $\widehat{zz}/p$. | $\widehat{\rho\rho}/p$. | $\widehat{zz}/p$. |
| ° | | | | | | | | |
| 0 | 0·75 | 1 | 0·75 | 1 | 0·75 | 1 | 0·75 | 1 |
| 10 | 0·634 | 0·999 | 0·615 | 0·998 | 0·589 | 0·998 | 0·560 | 0·996 |
| 20 | 0·525 | 0·990 | 0·490 | 0·987 | 0·440 | 0·982 | 0·387 | 0·972 |
| 30 | 0·430 | 0·968 | 0·382 | 0·959 | 0·313 | 0·941 | 0·243 | 0·912 |
| 40 | 0·355 | 0·927 | 0·298 | 0·909 | 0·213 | 0·872 | 0·135 | 0·812 |
| 50 | 0·300 | 0·866 | 0·235 | 0·836 | 0·144 | 0·777 | 0·064 | 0·679 |
| 60 | 0·266 | 0·786 | 0·197 | 0·743 | 0·102 | 0·661 | 0·025 | 0·529 |
| 70 | 0·249 | 0·690 | 0·178 | 0·636 | 0·080 | 0·538 | 0·009 | 0·382 |
| 80 | 0·245 | 0·583 | 0·172 | 0·523 | 0·073 | 0·416 | 0·006 | 0·256 |
| 90 | 0·247 | 0·472 | 0·173 | 0·411 | 0·073 | 0·307 | 0·007 | 0·160 |
| 100 | 0·249 | 0·363 | 0·176 | 0·307 | 0·075 | 0·215 | 0·012 | 0·095 |
| 110 | 0·245 | 0·264 | 0·174 | 0·217 | 0·075 | 0·143 | 0·015 | 0·054 |
| 120 | 0·229 | 0·178 | 0·165 | 0·142 | 0·072 | 0·088 | 0·014 | 0·030 |
| 130 | 0·199 | 0·109 | 0·146 | 0·085 | 0·065 | 0·050 | 0·012 | 0·015 |
| 140 | 0·151 | 0·058 | 0·116 | 0·044 | 0·053 | 0·025 | 0·011 | 0·007 |
| 150 | 0·088 | 0·025 | 0·077 | 0·019 | 0·037 | 0·010 | 0·008 | 0·003 |
| 160 | 0·009 | 0·008 | 0·029 | 0·006 | 0·018 | 0·003 | 0·004 | 0·001 |
| 170 | 0·080 | 0·001 | 0·027 | 0·001 | 0·004 | 0·000 | 0·000 | 0·000 |
| 180 | 0·176 | 0 | 0·087 | 0 | 0·028 | 0 | 0·004 | 0 |

TABLE III.

| $\phi$. | $k = \sin 5°$. | | $k = \sin 15°$. | | $k = \sin 30°$. | | $k = \sin 50°$. | |
|---|---|---|---|---|---|---|---|---|
| | $\widehat{\rho z}/p$. | $\widehat{\omega\omega}/p$. | $\widehat{\rho z}/p$. | $\widehat{\omega\omega}/p$. | $\widehat{\rho z}/p$. | $\widehat{\omega\omega}/p$. | $\widehat{\rho z}/p$. | $\widehat{\omega\omega}/p$. |
| ° 0 | 0 | 0·75 | 0 | 0·75 | 0 | 0·75 | 0 | 0·75 |
| 10 | 0·010 | 0·680 | 0·010 | 0·639 | 0·008 | 0·597 | 0·005 | 0·564 |
| 20 | 0·039 | 0·613 | 0·039 | 0·530 | 0·032 | 0·453 | 0·016 | 0·389 |
| 30 | 0·083 | 0·548 | 0·081 | 0·431 | 0·065 | 0·326 | 0·035 | 0·244 |
| 40 | 0·136 | 0·487 | 0·130 | 0·341 | 0·102 | 0·223 | 0·053 | 0·144 |
| 50 | 0·191 | 0·431 | 0·179 | 0·268 | 0·135 | 0·141 | 0·067 | 0·061 |
| 60 | 0·241 | 0·381 | 0·220 | 0·206 | 0·160 | 0·084 | 0·074 | 0·017 |
| 70 | 0·280 | 0·336 | 0·248 | 0·158 | 0·172 | 0·046 | 0·074 | 0·002 |
| 80 | 0·303 | 0·298 | 0·260 | 0·121 | 0·172 | 0·024 | 0·067 | 0·008 |
| 90 | 0·308 | 0·266 | 0·256 | 0·095 | 0·160 | 0·012 | 0·056 | 0·006 |
| 100 | 0·294 | 0·249 | 0·237 | 0·077 | 0·141 | 0·006 | 0·044 | 0·006 |
| 110 | 0·264 | 0·221 | 0·207 | 0·066 | 0·116 | 0·005 | 0·033 | 0·005 |
| 120 | 0·221 | 0·201 | 0·168 | 0·060 | 0·090 | 0·006 | 0·024 | 0·002 |
| 130 | 0·171 | 0·189 | 0·127 | 0·058 | 0·065 | 0·008 | 0·016 | 0·001 |
| 140 | 0·119 | 0·180 | 0·086 | 0·059 | 0·043 | 0·011 | 0·010 | 0·001 |
| 150 | 0·071 | 0·175 | 0·051 | 0·062 | 0·025 | 0·014 | 0·006 | 0·001 |
| 160 | 0·033 | 0·173 | 0·023 | 0·068 | 0·011 | 0·018 | 0·003 | 0·002 |
| 170 | 0·009 | 0·174 | 0·006 | 0·076 | 0·003 | 0·022 | 0·001 | 0·003 |
| 180 | 0 | 0·176 | 0 | 0·087 | 0 | 0·028 | 0 | 0·004 |

From these results it follows that, when $\sigma = \frac{1}{4}$, and $\rho = 0$,

$$\left.\begin{aligned}\frac{\widehat{\rho\rho}}{p} = \frac{\widehat{\omega\omega}}{p} &= \tfrac{1}{4}\left\{-3 + \frac{3z}{\sqrt{(a^2+z^2)}} + \frac{2a^2z}{(a^2+z^2)^{3/2}}\right\},\\ \frac{\widehat{zz}}{p} &= -1 + \frac{z}{\sqrt{(a^2+z^2)}} - \frac{a^2z}{(a^2+z^2)^{3/2}}, \quad \widehat{\rho z} = 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It appears that $\widehat{\rho\rho}$ and $\widehat{\omega\omega}$ are positive if $z/\sqrt{(a^2+z^2)}$ lies between 1 and $\frac{1}{2}(\sqrt{7}-1)$, *i.e.*, if $z/a > 1{\cdot}448$ approximately. Their maxima are found at $z/a = \sqrt{5}$, or $z/a = 2{\cdot}236$ approximately, and the common maximum of their values on the axis of $z$ is $(0{\cdot}0109)\,p$ approximately.

As a point P travels round a very small semicircle very close to the point $M_1$, $(k \to 0)$, from a place where $\phi = 0°$ to a place where $\phi = 180°$, the value of the stress-component $\widehat{\rho\rho}$ continually increases algebraically from $-(0{\cdot}75)\,p$ to $(0{\cdot}25)\,p$, changing sign when $\phi = 156°\ 10'\ 30''$ very nearly. As P travels round any of the selected semicircles, $k =$ constant, the value of $\widehat{\rho\rho}$ increases algebraically with some slight fluctuation from $-(0{\cdot}75)\,p$ to zero, vanishes and changes sign when $\phi$ has a certain value, which increases towards 180° as $k \to 1$, and thereafter increases to a certain positive value, which is attained when $\phi = 180°$, and this value diminishes towards zero as $k$ increases to 1. As P travels down the axis of $z$ from the centre of the pressed area, the value of $\widehat{\rho\rho}$ continually increases algebraically from the value $-(0\cdot75)\,p$ to the positive maximum specified above, and thereafter diminishes continually to zero.

On any of the semicircles $\widehat{zz}$ continually increases algebraically from the value $-p$ to zero, and $\widehat{\rho z}$ diminishes algebraically from the value zero to a negative minimum, thereafter increasing algebraically to zero. The minimum increases algebraically from $-p/\pi$, when $k \to 0$, to zero, when $k \to 1$. On the axis of $z$, $\widehat{\rho z}$ is constantly zero, and $\widehat{zz}$ behaves in the same way as on any of the semicircles.

As P travels round a very small semicircle very close to the point $M_1$ the value of $\widehat{\omega\omega}$ at P continually increases algebraically from the value $-(0\cdot75)\,p$ to the value $-(0\cdot25)\,p$. As P travels round a semicircle of the family $k =$ constant that is not very remote from $M_1$, $\widehat{\omega\omega}$ increases algebraically from the value $-(0\cdot75)\,p$ to a negative maximum, and then diminishes to a negative value, which decreases absolutely as $k$ increases. When the circle is more remote from $M_1$ the maximum is positive and quite small, but the final value is always negative. On the axis of $z$ the maximum has the positive value previously specified, and the final value is zero.

§ 6.3. The stress-component $\widehat{\omega\omega}$ is always a principal stress, and the axial plane through P is always a principal plane of the stress at P. The remaining principal planes of stress at P are at right angles to this plane, and the corresponding principal stresses are the quantities $T_1$ and $T_2$, given by the equations

$$T_1 = \tfrac{1}{2}(\widehat{\rho\rho} + \widehat{zz} + S), \qquad T_2 = \tfrac{1}{2}(\widehat{\rho\rho} + \widehat{zz} - S), \quad \ldots\ldots \quad (1)$$

$$S^2 = (\widehat{\rho\rho} - \widehat{zz})^2 + 4\widehat{\rho z}^2, \quad \ldots\ldots\ldots\ldots \quad (2)$$

and the condition that S is positive. Table IV shows the approximate values of $T_1/p$ and $T_2/p$ at the selected set of points. The signs are omitted. The values of $T_1$ below the thick horizontal lines are positive, those above them are negative. All the values of $T_2$ are negative.

As a point P travels round a semicircle $k =$ constant from a point where $\phi = 0°$ to a point where $\phi = 180°$, or down the axis of $z$, the value of $T_2$ at P continually increases algebraically from $-p$ to zero.

As P travels round a very small semicircle ($k \to 0$) the value of $T_1$ at P continually increases algebraically from $-(0\cdot75)\,p$ to $(0\cdot25)\,p$. It changes sign near the radius specified by $\phi = 100°$, and has a stationary value near the radius specified by $\phi = 140°$, without however attaining a maximum or a minimum. On any of the selected semicircles its value increases algebraically from $-(0\cdot75)\,p$ to a positive value, which diminishes as $k$ increases, and it fluctuates a little on the arc on which it is positive. Its value at $\phi = 180°$ is always its greatest value on the semicircle. As P travels down the axis of $z$ the value of $T_1$ is the same as that of $\widehat{\rho\rho}$, and its behaviour has been described already.

TABLE IV.

| $\phi$. | $k = \sin 5°$. | | $k = \sin 15°$. | | $k = \sin 30°$. | | $k = \sin 50°$. | |
|---|---|---|---|---|---|---|---|---|
| | $T_1/p$. | $T_2/p$. | $T_1/p$. | $T_2/p$. | $T_1/p$. | $T_2/p$. | $T_1/p$. | $T_2/p$. |
| 0° | 0·75 | 1 | 0·75 | 1 | 0·75 | 1 | 0·75 | 1 |
| 10 | 0·634 | 1·000 | 0·615 | 0·999 | 0·589 | 0·998 | 0·559 | 0·995 |
| 20 | 0·522 | 0·993 | 0·487 | 0·990 | 0·438 | 0·983 | 0·386 | 0·973 |
| 30 | 0·418 | 0·980 | 0·371 | 0·970 | 0·306 | 0·948 | 0·241 | 0·914 |
| 40 | 0·324 | 0·957 | 0·272 | 0·935 | 0·198 | 0·888 | 0·126 | 0·812 |
| 50 | 0·241 | 0·924 | 0·186 | 0·885 | 0·117 | 0·804 | 0·057 | 0·687 |
| 60 | 0·171 | 0·880 | 0·119 | 0·820 | 0·059 | 0·704 | 0·014 | 0·540 |
| 70 | 0·113 | 0·826 | 0·069 | 0·745 | 0·023 | 0·595 | 0·005 | 0·396 |
| 80 | 0·067 | 0·761 | 0·034 | 0·662 | 0·002 | 0·488 | 0·011 | 0·273 |
| 90 | 0·032 | 0·687 | 0·010 | 0·575 | 0·008 | 0·388 | 0·016 | 0·183 |
| 100 | 0·007 | 0·606 | 0·005 | 0·488 | 0·012 | 0·302 | 0·008 | 0·114 |
| 110 | 0·012 | 0·516 | 0·012 | 0·403 | 0·012 | 0·230 | 0·004 | 0·074 |
| 120 | 0·019 | 0·426 | 0·015 | 0·322 | 0·010 | 0·171 | 0·004 | 0·047 |
| 130 | 0·023 | 0·330 | 0·015 | 0·245 | 0·008 | 0·123 | 0·003 | 0·030 |
| 140 | 0·023 | 0·233 | 0·013 | 0·174 | 0·006 | 0·084 | 0·001 | 0·019 |
| 150 | 0·021 | 0·134 | 0·011 | 0·106 | 0·004 | 0·052 | 0·001 | 0·011 |
| 160 | 0·025 | 0·042 | 0·009 | 0·043 | 0·003 | 0·024 | 0·001 | 0·006 |
| 170 | 0·081 | 0·002 | 0·027 | 0·001 | 0·005 | 0·002 | 0·001 | 0·001 |
| 180 | 0·176 | 0 | 0·087 | 0 | 0·028 | 0 | 0·004 | 0 |

§ 6.4. For a discussion of strength or weakness the most important quantities are the stress-difference and the tensile stress. The stress-difference is the excess of the algebraically greatest principal stress at a point above the algebraically smallest principal stress at the same point. It is generally conceded that failure is likely to occur if at any point the stress-difference exceeds some limit depending upon the material. In the present case the stress-difference is either S, which is $T_1 - T_2$, or else it may be $T_1 - \widehat{\omega\omega}$. The approximate values of $S/p$ for the selected set of points are shown in Table V. All the values are positive.

As a point P travels round a very small semicircle ($k \to 0$) from a point where $\phi = 0°$ to a point where $\phi = 180°$ the value of S increases from $(0{\cdot}25)\,p$ to

$$\sqrt{\left\{\frac{1}{16} + \frac{2}{\pi^2} + \frac{\sqrt{(16+\pi^2)}}{2\pi^2}\right\}}\, p, \qquad (1)$$

which is approximately $(0{\cdot}723)\,p$. This maximum value is attained when $\phi$ has the smallest positive value which satisfies the equation

$$4 \tan 2\phi = -\pi, \qquad (2)$$

and this value of $\phi$ is approximately 71°. Thereafter the value of S continually diminishes until $\phi$ is equal to the second smallest positive value which satisfies the same

TABLE V.—(S/$p$).

| $\phi$ \ $k$. | sin 5°. | sin 15°. | sin 30°. | sin 50°. |
|---|---|---|---|---|
| ° | | | | |
| 0 | 0·25 | 0·25 | 0·25 | 0·25 |
| 10 | 0·366 | 0·384 | 0·409 | 0·436 |
| 20 | 0·472 | 0·503 | 0·546 | 0·586 |
| 30 | 0·562 | 0·599 | 0·642 | 0·673 |
| 40 | 0·633 | 0·663 | 0·690 | 0·686 |
| 50 | 0·683 | 0·699 | 0·688 | 0·630 |
| 60 | 0·709 | 0·701 | 0·645 | 0·525 |
| 70 | 0·712 | 0·676 | 0·573 | 0·401 |
| 80 | 0·694 | 0·628 | 0·486 | 0·283 |
| 90 | 0·655 | 0·565 | 0·397 | 0·199 |
| 100 | 0·599 | 0·493 | 0·314 | 0·122 |
| 110 | 0·528 | 0·415 | 0·242 | 0·077 |
| 120 | 0·444 | 0·337 | 0·181 | 0·051 |
| 130 | 0·353 | 0·261 | 0·132 | 0·033 |
| 140 | 0·255 | 0·187 | 0·091 | 0·028 |
| 150 | 0·156 | 0·117 | 0·057 | 0·012 |
| 160 | 0·066 | 0·052 | 0·027 | 0·006 |
| 170 | 0·083 | 0·028 | 0·007 | 0·001 |
| 180 | 0·176 | 0·087 | 0·028 | 0·004 |

equation, and this value is approximately 161°. Then the value of S increases to $(0 \cdot 25)p$, to which it becomes equal when $\phi = 180°$. As P travels round any of the selected semicircles, the value of S increases from $(0 \cdot 25)\,p$ to a certain maximum, then diminishes to a minimum, and afterwards increases to a final value, which is less than $(0 \cdot 25)\,p$ and diminishes as $k$ increases. As P travels down the axis of $z$ the value of S increases from $(0 \cdot 25)\,p$ to a maximum value, which is found by putting

$$13z^2 = 5a^2, \quad z > 0, \qquad (3)$$

in the formula

$$\frac{S}{p} = \frac{1}{4}\left\{1 - \frac{z}{\sqrt{(a^2 + z^2)}} + \frac{6a^2 z}{(a^2 + z^2)^{3/2}}\right\}. \qquad (4)$$

This maximum value is approximately $(0 \cdot 689)\,p$, and is attained at a depth approximately equal to $(0 \cdot 620)\,a$. Table VI shows the approximate values of the maxima of

TABLE VI.—Maxima of S/$p$.

| | $k \to 0$. | $k = \sin 5°$. | $k = \sin 15°$. | $k = \sin 30°$. | $k = \sin 50°$. | $k \to 1$. |
|---|---|---|---|---|---|---|
| S/$p$ . . . . . | 0·723 | 0·714 | 0·704 | 0·695 | 0·690 | 0·689 |
| $\phi$ . . . . . . | 71° | 67° | 56° | 44° | 37° | 32° |
| $\rho/a$ . . . . . | 1 | 0·934 | 0·742 | 0·446 | 0·184 | 0 |
| $z/a$ . . . . . | 0 | 0·156 | 0·383 | 0·536 | 0·615 | 0·620 |

$S/p$, and (approximately) the positions at which they are attained. In every case the maximum value of S on a semicircle exceeds that of $T_1 - \widehat{\omega\omega}$.

§ 6.5. To interpret these results we revert to the case of a long rectangle discussed in § 2.6. In that case it appears that the stress-difference is constant along any circular arc, having its centre on the axis of $z$, and passing through the points $M_1$ and $M_2$. It has its greatest value when the arc is a semicircle, this greatest value is $(2/\pi)\,p$, or approximately $(0\cdot637)\,p$, and it is attained at every point of the semicircle.* This result indicates that the foundation under the middle part of a long wall would be most likely to give way at some point on a cylindrical surface under the wall, the transverse section of the surface being a semicircle, and that it would be as likely to give way at one point on the surface as at another.

In the case of a circular area subjected to uniform pressure there is not strictly a surface over which the stress-difference is a maximum and constant. But there is a locus which has a roughly similar property. As a point P travels round a semicircle $k =$ constant, the stress-difference passes through a maximum value; the surface in question is the locus of these partial maxima. The stress-difference varies very slowly along this locus, increasing slightly all the way from the axis to the circular edge. The solution in Chapter 5 suggests that, if the pressure is not strictly uniform, but tends to zero at the circular edge of the pressed area, the variations in the stress-difference on the locus of partial maxima are likely to be smaller than they are for uniform pressure. For, in the case of a hemispherical distribution of pressure, the stress-difference decreases from axis to edge.

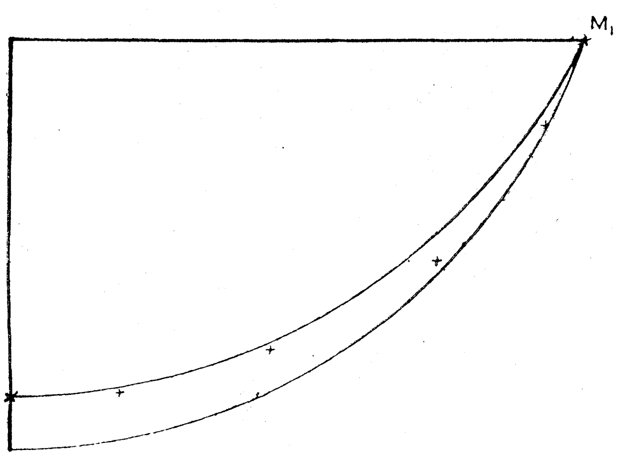


Fig. 6.

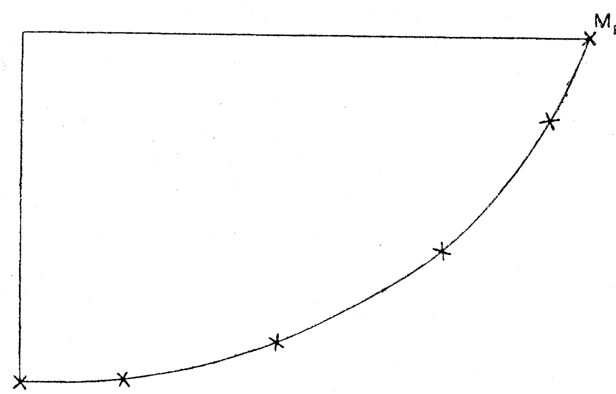


Fig. 7.

The locus of partial maxima of the stress-difference for uniform pressure over a circular area is a basin-shaped surface of revolution about the axis of the circle. It passes through the circle, and lies between two segments of spheres, which have their centres on the axis, and pass through the circle. These spheres cut the axis at depths equal to

* I have to thank Prof. C. F. Jenkin for calling my attention to the importance of this result and suggesting that it might be extended to forms of the pressed area other than the long rectangle.

0·620 and 0·712 times the radius of the circle, and the locus in question touches the former at the point where it cuts the axis, and the latter along the circle.

In fig. 6 the two circular arcs are the traces of the two spherical segments on an axial plane, and the points marked with crosses are the points of the locus which have been determined. Fig. 7 is a rough drawing of the meridian section of the locus.

It appears to be reasonable to conclude that the foundation under a round pillar would be most likely to give way at some point of such a basin-shaped surface as has been described, and that it would be nearly as likely to give way at one point of the surface as at another.

§ 6.6. It is usual to regard a large tensile stress as a source of weakness, whether it is accompanied by a large stress-difference or not. In the case of uniform pressure over a circular area, the greatest tensile stress is the principal stress $T_1$, at points just outside the pressed area, and on the initially plane boundary of the solid. The limiting value for $z = 0$ and $\rho$ just greater than $a$ is $(0 \cdot 25)\ p$. At the same places there is a second principal stress $\widehat{\omega\omega}$, which has the limiting value $-(0 \cdot 25)\ p$, so that there is a stress-difference $(0 \cdot 5)p$. A greater stress-difference of type S is found however in the immediate neighbourhood of the point. If the pressure is not strictly uniform, but tends to zero at the boundary of the pressed area, the values of $T_1$ and $\widehat{\omega\omega}$ are unaltered, but the greater value of S in the neighbourhood does not present itself. It would seem that the foundation under a round pillar is in some danger of failing through the occurrence of high tensile stress, and that the region of weakness would be near the base of the pillar, and just outside it. It would also seem that the danger arising from this cause is not extreme. But the theory, which has been given, for uniform and varying pressure over a rectangular area, suggests that there may be considerable danger of failure from this cause wherever the boundary of the pressed area presents a sharp corner. It was shown that, if the pressure over a rectangular area were strictly uniform, the calculated tensile stress would be infinite, and, although the infinity can be removed by supposing that the pressure tends to zero at the boundary of the pressed area, it would remain rather high if the pressure were nearly uniform.